

ESAT

Maths 1

Indices, surds and bounds

Free lesson notes

Covers	M2.7, M2.11 to M2.13
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Indices and surds reward fluency. The rules are few, but the ESAT combines them: a fractional index inside a fraction, a surd that must be rationalised before it can be added, a bound that depends on which end of an interval you take. With no calculator and about ninety seconds a question, the students who do well are those who know *why* each rule is true, because then they can rebuild it under pressure and are not fooled by a look-alike.

This lesson builds the three topics from the ground up. We start with what a power *means* and derive the laws of indices by counting; extend them to zero, negative and fractional indices; and use them to solve equations and simplify expressions. Next come surds: why $\sqrt{2}$ is not a fraction, how to simplify and combine surds, how to expand brackets that contain them, and how to rationalise a denominator. Finally we study rounding, error intervals and upper and lower bounds, including the rule for which bound to use in each kind of calculation.

How to use it. Read each idea, then cover the worked example and try it yourself before reading the solution. Redraw the diagrams: most of the rules are easy to remember once you have seen why. Finish with the self-check, then the practice set. Specification codes: M2.7 (index laws), M2.11 (surds), M2.12 (bounds) and M2.13 (rounding and error intervals).

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1 Powers from meaning: the first three laws

What a power means. Writing a^n is shorthand for multiplying n copies of a together: $2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$ and $10^3 = 10 \times 10 \times 10 = 1000$. The number a is the *base* and n is the *index* (also called the power or exponent). At this stage the index is a whole number counting copies. Every law of indices in this lesson follows from this one meaning, and from nothing else.

Law 1: multiplying powers of the same base. To find $a^3 \times a^2$ write out the copies. We have three copies of a times two more copies, which is five copies in all, so $a^3 \times a^2 = a^5 = a^{3+2}$. The picture shows the counting. The rule is: *when you multiply, add the indices*.

$$\underbrace{a \times a \times a}_{a^3: \text{ three factors }} \times \underbrace{a \times a}_{a^2: \text{ two factors }} = a^5, \text{ so } a^3 \times a^2 = a^{3+2}$$

$$\frac{a \times a \times a \times a \times a}{a \times a} = \frac{a \times a \times a \times \cancel{a} \times \cancel{a}}{\cancel{a} \times \cancel{a}} = a^3, \text{ so } a^5 \div a^2 = a^{5-2}$$

$$(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6, \text{ so } (a^2)^3 = a^{2 \times 3}$$

Law 2: dividing powers of the same base. In the middle line of the picture, $a^5 \div a^2$ is five copies over two copies. Each copy on the bottom cancels one on the top, leaving three: $a^5 \div a^2 = a^3 = a^{5-2}$. *When you divide, subtract the indices.*

Law 3: a power of a power. The bracket $(a^2)^3$ means three copies of a^2 multiplied: $a^2 \times a^2 \times a^2$, which by Law 1 is $a^{2+2+2} = a^6$. Three lots of two is 3×2 , so $(a^2)^3 = a^{2 \times 3}$. *A power of a power multiplies the indices.* The same idea gives $(ab)^n = a^n b^n$ and $(\frac{a}{b})^n = \frac{a^n}{b^n}$, because every factor inside the bracket is repeated n times.

The first index laws

For any base a and whole numbers m, n :

$$a^m \times a^n = a^{m+n} \quad a^m \div a^n = a^{m-n} \quad (a^m)^n = a^{mn} \quad (ab)^n = a^n b^n$$

These work *only when the bases are the same* (Laws 1 and 2) and they apply to *multiplying and dividing*, never to

adding and subtracting.

Worked example: simplifying with several letters

Simplify $\frac{(3x^2y)^3}{9x^4y}$.

Cube the bracket: $(3x^2y)^3 = 3^3 x^{2 \times 3} y^3 = 27x^6y^3$. Divide number by number and each letter by itself: $\frac{27}{9} = 3$, $x^{6-4} = x^2$, and $y^{3-1} = y^2$. The answer is $3x^2y^2$.

Worked example: a power of a product

Write 12^3 in the form $2^a \times 3^b$.

$12 = 2^2 \times 3$, so $12^3 = (2^2 \times 3)^3 = (2^2)^3 \times 3^3 = 2^6 \times 3^3$. (Check: $12^3 = 1728$ and $64 \times 27 = 1728$.)

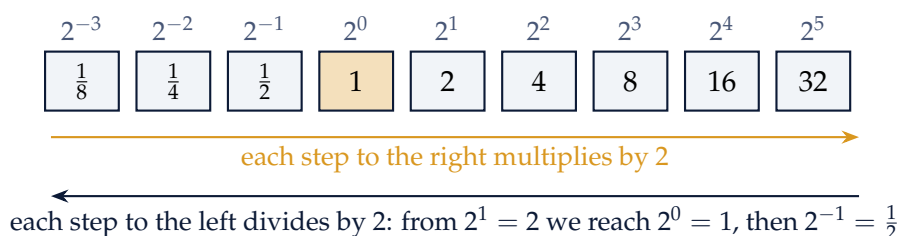
Common mistakes

- Adding indices when the terms are *added*: $2^3 + 2^4 \neq 2^7$. Evaluate instead: $8 + 16 = 24$.
- Multiplying the bases: $2^3 \times 2^4 = 2^7$, not 4^7 . The base stays the same; only the index changes.
- Using Law 1 on different bases: $2^3 \times 3^2$ cannot be written as a single power. It is just $8 \times 9 = 72$.
- Squaring a sum: $(a + b)^2 \neq a^2 + b^2$. The bracket law $(ab)^n = a^n b^n$ is for *products*.

2 Zero, negative and fractional indices

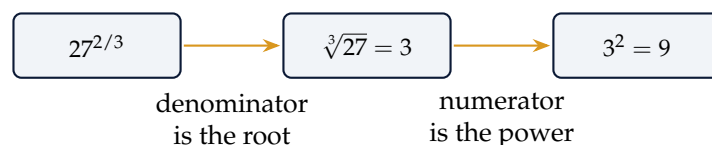
So far an index counts copies, so it makes no sense to ask for “zero copies” or “minus one copy”. Mathematicians therefore *extend* the meaning so that the laws above keep working. The extension is forced on us, as we shall see.

The pattern. Look at the powers of 2. Each step to the right multiplies by 2, so each step to the left must divide by 2. Keep going left past $2^1 = 2$ and the pattern gives $2^0 = 1$, then $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$, and so on.



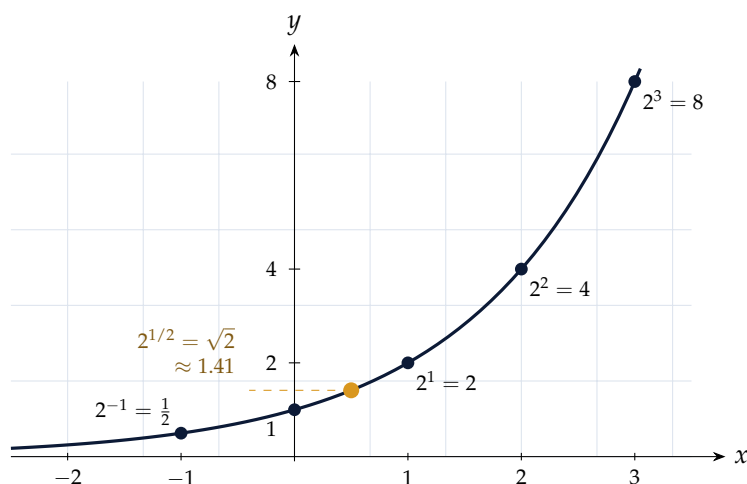
Why the pattern is not a choice. Law 2 says $a^3 \div a^3 = a^{3-3} = a^0$. But anything (other than 0) divided by itself is 1. So we must have $a^0 = 1$. In the same way $a^2 \div a^5 = a^{-3}$, while cancelling copies gives $\frac{a \times a}{a \times a \times a \times a \times a} = \frac{1}{a^3}$. So $a^{-3} = \frac{1}{a^3}$. A *negative index means “reciprocal”*: it does not make the number negative. For a fraction, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$: flip the fraction and the index becomes positive.

Fractional indices. What should $a^{1/2}$ mean? If Law 3 is to hold, $\left(a^{1/2}\right)^2 = a^{\frac{1}{2} \times 2} = a^1 = a$. So $a^{1/2}$ is a number which, squared, gives a : the square root. Similarly $a^{1/3}$ cubed must give a , so it is the cube root, and in general $a^{1/n} = \sqrt[n]{a}$. For $a^{m/n}$ use Law 3 again: $a^{m/n} = \left(a^{1/n}\right)^m$. The denominator of the fraction is the *root* and the numerator is the *power*.



For a fractional index *take the root first*, then the power, so that the numbers stay small: $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$, which is far easier than $\sqrt[3]{27^2} = \sqrt[3]{729}$.

A picture. The graph of $y = 2^x$ passes through all the whole-number powers and also through every fractional one: at $x = \frac{1}{2}$ it passes through $\sqrt{2} \approx 1.41$, between $2^0 = 1$ and $2^1 = 2$ as it should. A power with a positive base is *always* positive, however negative the index, and the curve never touches the x -axis.



Zero, negative and fractional indices

$$a^0 = 1 \quad a^{-n} = \frac{1}{a^n} \quad a^{1/n} = \sqrt[n]{a} \quad a^{m/n} = (\sqrt[n]{a})^m \quad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

Useful values to know: $2^{10} = 1024 \approx 10^3$; squares to 15^2 ; cubes $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, $5^3 = 125$; fourth powers $2^4 = 16$, $3^4 = 81$.

Worked example: fractional and negative

Evaluate $\left(\frac{8}{27}\right)^{-2/3}$.

The negative index flips the fraction: $\left(\frac{27}{8}\right)^{2/3}$. Roots first: $\sqrt[3]{27} = 3$ and $\sqrt[3]{8} = 2$. Then square: $\left(\frac{3}{2}\right)^2 = \frac{9}{4}$.

Worked example: mixing the rules

Evaluate $\frac{16^{3/4} \times 2^{-2}}{8^{-1/3}}$.

$16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$. Also $2^{-2} = \frac{1}{4}$ and $8^{-1/3} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$. So the value is $\frac{8 \times \frac{1}{4}}{\frac{1}{2}} = \frac{2}{\frac{1}{2}} = 4$.

Common mistakes

- $a^{-n} = -a^n$: a negative index is a reciprocal, not a negative number. $2^{-3} = \frac{1}{8}$.
- $-2^{-2} = -\frac{1}{4}$ but $(-2)^{-2} = +\frac{1}{4}$: the index applies to the number next to it only, unless there is a bracket.
- Reading $27^{2/3}$ as $27 \times \frac{2}{3} = 18$. The index is not a multiplier: denominator means root, numerator means power.
- Believing $a^0 = 0$. It is 1 for every non-zero a .

3 Solving and simplifying with indices

Equal bases. Powers of the same base (> 0 , $\neq 1$) are equal only when the indices are equal. So to solve an equation with the unknown in an index, write both sides as powers of the *same* base and equate the indices. It helps to know the first few powers of the small primes: $4 = 2^2$, $8 = 2^3$, $16 = 2^4$, $32 = 2^5$, $9 = 3^2$, $27 = 3^3$, $81 = 3^4$, $25 = 5^2$.

Worked example: equal bases

Solve $4^x = 8^{x-1}$.

Write both with base 2: $(2^2)^x = (2^3)^{x-1}$, so $2^{2x} = 2^{3x-3}$. Equate the indices: $2x = 3x - 3$, so $x = 3$. (Check: $4^3 = 64 = 8^2$.)

Worked example: a harder equation

Solve $9^x \times 3 = 27^{2-x}$.

Base 3: $3^{2x} \times 3^1 = 3^{3(2-x)}$, so $3^{2x+1} = 3^{6-3x}$. Then $2x + 1 = 6 - 3x$, so $5x = 5$ and $x = 1$.

Factorising out a power. Powers cannot be added by adding their indices, but they can be *factorised*. For example $2^{n+3} - 2^{n+1}$: write $2^{n+3} = 2^n \times 2^3$ and $2^{n+1} = 2^n \times 2$, so the expression is $2^n(8 - 2) = 6 \times 2^n$. Always look for the smallest power and factor it out.

Worked example: factorising a sum of powers

Simplify $\frac{3^{n+2} + 3^n}{3^{n-1}}$.

Factor 3^n from the top: $3^{n+2} + 3^n = 3^n(9 + 1) = 10 \times 3^n$. Dividing by 3^{n-1} gives $10 \times 3^{n-(n-1)} = 10 \times 3 = 30$.

Worked example: a deceptive sum

Write $2^{50} + 2^{50}$ as a single power of 2.

There are two identical terms, so the sum is $2 \times 2^{50} = 2^1 \times 2^{50} = 2^{51}$. It is not 2^{100} (that would be a product, $2^{50} \times 2^{50}$).

Exam tip

Estimation: $2^{10} = 1024 \approx 10^3$. So $2^{20} \approx 10^6$ and $2^{30} \approx 10^9$. For the number of digits of 2^{40} , think $2^{40} = (2^{10})^4 \approx 10^{12}$.

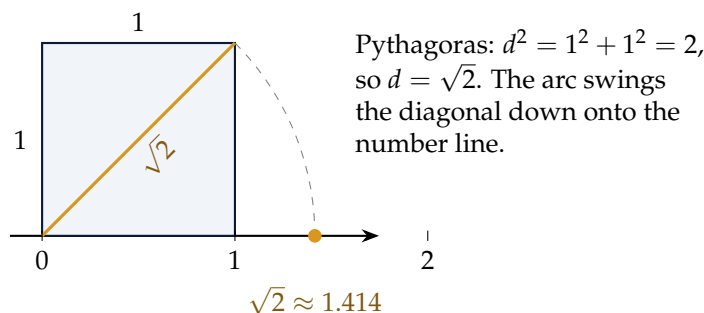
Common mistakes

- Turning $4^x = 8^{x-1}$ into $2x = 3x - 1$. The *whole* index $x - 1$ is multiplied by 3, giving $3x - 3$.
- Equating indices before the bases match: $4^x = 8^x$ has no non-zero solution, whatever you do with x .
- Dividing $2^{n+3} - 2^{n+1}$ term by term. Factorise, do not subtract indices.

4 Surds: roots that never end

Square roots that are not whole numbers. $\sqrt{9} = 3$ because $3^2 = 9$. But no whole number squares to 2, and no fraction does either. A number such as $\sqrt{2}$ that is a root but is *not* a fraction is called a *surd*; its decimal never ends and never repeats ($\sqrt{2} = 1.41421356\dots$). The word for such numbers is *irrational*. Because it is irrational, an exact answer such as $3\sqrt{2}$ is left in surd form, not rounded.

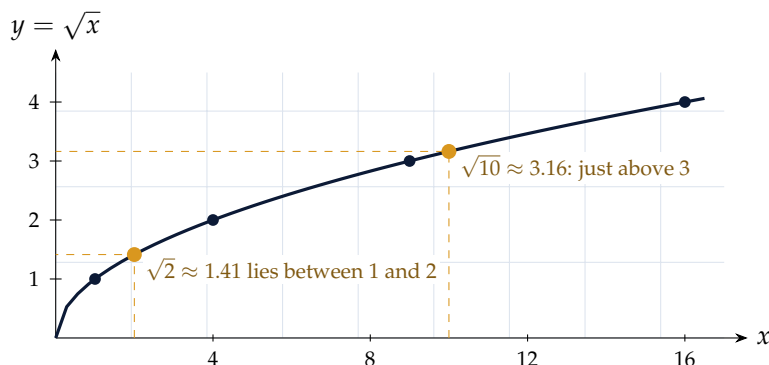
Where does $\sqrt{2}$ come from? Draw a square with sides of length 1. Its diagonal d satisfies, by Pythagoras, $d^2 = 1^2 + 1^2 = 2$, so the diagonal is exactly $\sqrt{2}$. Swinging that diagonal onto the number line shows that $\sqrt{2}$ is a perfectly good point, about 1.414.



Why $\sqrt{2}$ is not a fraction. Suppose $\sqrt{2} = \frac{p}{q}$ in lowest terms. Then $p^2 = 2q^2$, so p^2 is even and so p is even, say $p = 2k$.

Then $4k^2 = 2q^2$, so $q^2 = 2k^2$ and q is even too. But then p and q share the factor 2, contradicting “lowest terms”. So no such fraction exists.

Estimating a surd. Sandwich it between the squares on either side. Since $1^2 = 1 < 2 < 4 = 2^2$, $\sqrt{2}$ is between 1 and 2; since $3^2 = 9 < 10 < 16 = 4^2$, $\sqrt{10}$ is between 3 and 4, and as 10 is just above 9, the root is just above 3 (it is 3.16). The graph of $y = \sqrt{x}$ shows this: it climbs steeply at first and then flattens.



Two rules for roots of products.

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \quad (a, b \geq 0)$$

They hold because $(\sqrt{a} \times \sqrt{b})^2 = \sqrt{a}^2 \times \sqrt{b}^2 = ab$. They let us *simplify* a surd by pulling out the largest square factor: $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$, and $\sqrt{\frac{5}{16}} = \frac{\sqrt{5}}{4}$. In the other direction, $3\sqrt{2} = \sqrt{9} \times \sqrt{2} = \sqrt{18}$.

Adding and subtracting. Two surds can be combined only if they are *like* surds, with the same surd part, just as $6x + 3x = 9x$ but $6x + 3y$ cannot be simplified. So simplify each surd first, then collect.

$$\begin{aligned} \sqrt{72} &= 6\sqrt{2} && \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \\ \sqrt{18} &= 3\sqrt{2} && \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \\ \sqrt{72} + \sqrt{18} &= 9\sqrt{2} && \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \boxed{\sqrt{2}} \end{aligned}$$

adding like surds is
like adding like terms:
 $6x + 3x = 9x$

Worked example: simplify and collect

Simplify $\sqrt{72} - \sqrt{18} + \sqrt{50}$.

$\sqrt{72} = 6\sqrt{2}$, $\sqrt{18} = 3\sqrt{2}$ and $\sqrt{50} = 5\sqrt{2}$. So the sum is $(6 - 3 + 5)\sqrt{2} = 8\sqrt{2}$.

Worked example: multiplying surds

Simplify $\sqrt{12} \times \sqrt{6}$.

$\sqrt{12} \times \sqrt{6} = \sqrt{72} = 6\sqrt{2}$. (Alternatively $2\sqrt{3} \times \sqrt{6} = 2\sqrt{18} = 2 \times 3\sqrt{2} = 6\sqrt{2}$.)

Common mistakes

- $\sqrt{9 + 16} = 5$, not $\sqrt{9} + \sqrt{16} = 7$. Roots split over multiplication and division, never over addition.
- Simplifying $\sqrt{72}$ as $2\sqrt{18}$ and stopping. Take out the *largest* square factor (36) so that no square factor remains.
- Adding $\sqrt{2} + \sqrt{3} = \sqrt{5}$. Unlike surds cannot be combined (the decimals show it: $1.41 + 1.73 = 3.15$, while $\sqrt{5} = 2.24$).

5 Expanding brackets with surds

Brackets containing surds are expanded exactly as in algebra: every term in the first bracket multiplies every term in the second. The only new fact is that a surd times itself is a whole number: $\sqrt{a} \times \sqrt{a} = a$. A 2×2 multiplication grid keeps the four products organised.

On the left we expand $(2 + \sqrt{3})^2$. The two shaded cross terms are equal, which is why a square has a “middle term”: $(a + b)^2 = a^2 + 2ab + b^2$. On the right, the second bracket has a minus sign, and the cross terms are opposites that *cancel*.

	2	$\sqrt{3}$	
2	4	$2\sqrt{3}$	
$\sqrt{3}$	$2\sqrt{3}$	3	

$$(2 + \sqrt{3})^2 = 4 + 2\sqrt{3} + 2\sqrt{3} + 3 = 7 + 4\sqrt{3}$$

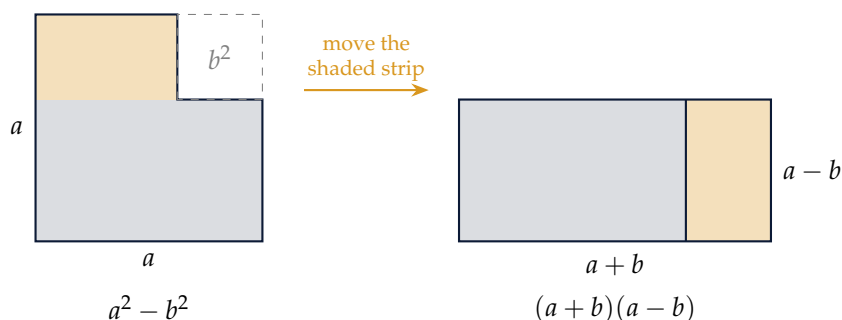
	2	$\sqrt{3}$	
2	4	$2\sqrt{3}$	
$-\sqrt{3}$	$-2\sqrt{3}$	-3	

$$(2 + \sqrt{3})(2 - \sqrt{3}) = 4 + 2\sqrt{3} - 2\sqrt{3} - 3 = 1: \text{the surds cancel}$$

This second case is so useful it has its own name. Brackets like $(p + \sqrt{q})$ and $(p - \sqrt{q})$ are *conjugates*, and their product is rational:

$$(p + \sqrt{q})(p - \sqrt{q}) = p^2 - q.$$

This is the *difference of two squares*, $(a + b)(a - b) = a^2 - b^2$, which can be seen as areas: a square of side a with a corner of side b removed has area $a^2 - b^2$, and cutting off the strip and moving it makes an $(a + b)$ by $(a - b)$ rectangle.



Worked example: expanding

Expand and simplify $(3 + \sqrt{5})(2 - \sqrt{5})$.

Multiply out: $3 \times 2 = 6$, $3 \times (-\sqrt{5}) = -3\sqrt{5}$, $\sqrt{5} \times 2 = 2\sqrt{5}$, $\sqrt{5} \times (-\sqrt{5}) = -5$. Collecting, $6 - 5 + (-3\sqrt{5} + 2\sqrt{5}) = 1 - \sqrt{5}$.

Worked example: a square with a surd inside

Write $\sqrt{7 + 4\sqrt{3}}$ in the form $a + b\sqrt{3}$ with a, b whole numbers.

If $\sqrt{7 + 4\sqrt{3}} = a + b\sqrt{3}$ then squaring, $7 + 4\sqrt{3} = a^2 + 3b^2 + 2ab\sqrt{3}$. Matching the whole-number and surd parts: $a^2 + 3b^2 = 7$ and $2ab = 4$. Trying $a = 2, b = 1$ works for both. So $\sqrt{7 + 4\sqrt{3}} = 2 + \sqrt{3}$. Check: $(2 + \sqrt{3})^2 = 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3}$.

Worked example: a length

A right-angled triangle has perpendicular sides $\sqrt{12}$ and $\sqrt{27}$. Find its area.

$\sqrt{12} = 2\sqrt{3}$ and $\sqrt{27} = 3\sqrt{3}$. Area = $\frac{1}{2} \times 2\sqrt{3} \times 3\sqrt{3} = \frac{1}{2} \times 6 \times 3 = 9$. (The area is a whole number: surds multiplied by themselves lose the root.)

Common mistakes

- Squaring a bracket term by term: $(2 + \sqrt{3})^2 \neq 4 + 3$. There is a middle term $2 \times 2 \times \sqrt{3}$.
- Writing $(\sqrt{3})^2 = \sqrt{9} = 3$ is fine, but $\sqrt{3} \times \sqrt{3}$ is 3, not $\sqrt{3}$. A surd times itself is a whole number.
- Forgetting the sign in $(p + \sqrt{q})(p - \sqrt{q}) = p^2 - q$: it is a *subtraction*.

6 Rationalising the denominator

A fraction with a surd on the bottom, such as $\frac{1}{\sqrt{2}}$, is awkward to use. By convention we write it with a *rational* denominator. You can see the benefit when estimating: $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = 0.707$ is an easy division, while $1 \div 1.414$ is not.

A single surd. Multiply top and bottom by that surd. The fraction keeps the same value (we multiplied by $\frac{\sqrt{2}}{\sqrt{2}} = 1$) and the denominator becomes whole: $\frac{6}{\sqrt{3}} = \frac{6\sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$.

A denominator such as $p + \sqrt{q}$. Multiplying by $\sqrt{q}$ does not help (try it: you are left with $\sqrt{q}$ terms). Instead multiply top and bottom by the *conjugate* $p - \sqrt{q}$. By the difference of two squares the new denominator is $p^2 - q$, with no root at all.

Worked example: using the conjugate

Write $\frac{10}{\sqrt{5} - 1}$ with a rational denominator.

The conjugate of $\sqrt{5} - 1$ is $\sqrt{5} + 1$. Multiply top and bottom by it: the denominator becomes $(\sqrt{5})^2 - 1^2 = 5 - 1 = 4$.

So $\frac{10(\sqrt{5} + 1)}{4} = \frac{5(\sqrt{5} + 1)}{2}$.

Worked example: a sum of conjugate fractions

Evaluate $\frac{1}{2 + \sqrt{3}} + \frac{1}{2 - \sqrt{3}}$.

Do not rationalise each one separately. Use the common denominator $(2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1$. The numerator is $(2 - \sqrt{3}) + (2 + \sqrt{3}) = 4$. So the value is 4.

Worked example: the reciprocal of a surd sum

Given $x = \sqrt{3} + \sqrt{2}$, find $x^2 + \frac{1}{x^2}$.

$\frac{1}{x} = \frac{1}{\sqrt{3} + \sqrt{2}} \times \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \frac{\sqrt{3} - \sqrt{2}}{3 - 2} = \sqrt{3} - \sqrt{2}$. Then $x^2 = 5 + 2\sqrt{6}$ and $\frac{1}{x^2} = 5 - 2\sqrt{6}$, so the sum is 10.

Common mistakes

- Multiplying only the denominator. Whatever you multiply the bottom by, multiply the top by the same.
- Using $\sqrt{5} + 1$ as the multiplier for a denominator of $\sqrt{5} - 1$: that gives $(\sqrt{5} + 1)^2$, which still contains a surd. *Change the sign.*
- Leaving an answer such as $\frac{2}{\sqrt{3}}$ in a question that asks for “a rational denominator”.

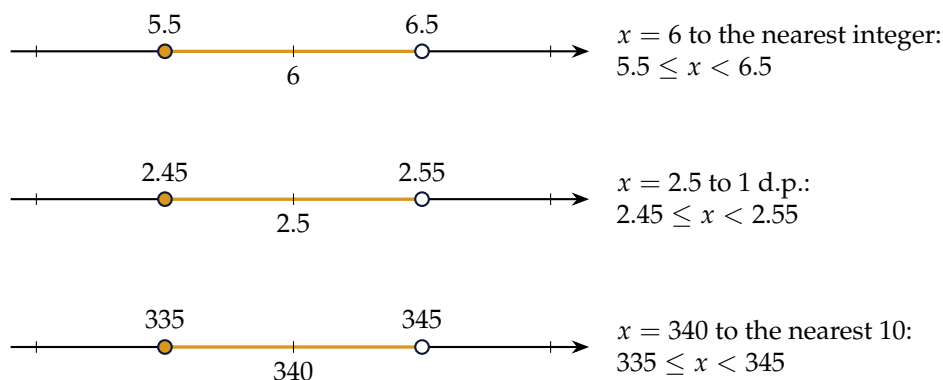
7 Rounding and error intervals

Rounding. A measured or rounded value is not exact: it stands for a whole *interval* of possible true values. If $x = 6$ to the nearest integer, then x could be anything that rounds to 6: from 5.5 up to (but not including) 6.5. We write the *error interval*

$$5.5 \leq x < 6.5.$$

The lower end 5.5 is included because it rounds *up* to 6. The upper end 6.5 is excluded because it rounds up to 7. The picture shows three cases. In general, if a number is rounded to the nearest u , the true value lies within *half a unit* on either side:

$$\text{value} - \frac{u}{2} \leq x < \text{value} + \frac{u}{2}.$$



The left end is the **lower bound** and the right end is the **upper bound**. Although the true value can never equal the upper bound, we still use it as the upper bound in calculations, because values can get as close to it as we like.

Decimal places and significant figures. To 1 d.p. the unit is 0.1, so the half-unit is 0.05 and 2.5 has the error interval $2.45 \leq x < 2.55$. To the nearest 10 the half-unit is 5. To 2 significant figures, 4.7 is $4.65 \leq x < 4.75$, and 3800 to 2 s.f. is $3750 \leq x < 3850$: the second significant figure of 3800 is the hundreds digit, so the unit is 100.

Truncation is different. If we simply *chop off* the later digits without rounding, then 3.4 means $3.4 \leq x < 3.5$: all the values whose first decimal is 4. There is no half-unit shift. A value truncated to the nearest unit below always lies in $[v, v + u)$.

Worked example: error interval

A mass is 2.35 kg to 2 decimal places. Write the error interval and say what the lower and upper bounds are. The unit is 0.01, so the half-unit is 0.005. The interval is $2.345 \leq m < 2.355$. The lower bound is 2.345 and the upper bound is 2.355.

Worked example: a different unit

A town's population is given as 48 000 to 2 significant figures. What is the greatest number of people it could have? The unit is 1000 (the second significant figure is the thousands digit), so the interval is $47\,500 \leq p < 48\,500$. The population could be at most 48 499 (the upper bound 48 500 is not reached; as a bound it is 48 500).

Common mistakes

- Writing 2.5 (to 1 d.p.) as $2.4 \leq x < 2.6$. The half-unit is 0.05, not 0.1: $2.45 \leq x < 2.55$.
- Getting the inequality signs the wrong way: the *lower* bound is included ($\leq$) and the *upper* is not ($<$).
- Treating truncation like rounding. A value truncated to 3.4 lies in $3.4 \leq x < 3.5$.

8 Calculating with bounds

When rounded values are put into a calculation, the answer is also uncertain. To find its greatest and least possible values, think about *each ingredient separately*: which end of its interval makes the answer as big (or as small) as possible?

Sums and products. Bigger inputs make a bigger sum or product, so use all the upper bounds for the greatest value and all the lower bounds for the least value.

Differences and quotients. Here one of the inputs works *against* you. For $a - b$ the answer is greatest when a is as large as possible and b as *small* as possible, and least when a is as small as possible and b as *large* as possible. For $\frac{a}{b}$ the answer is greatest when a is large and b is small, since dividing by a small number gives a big result.

Which bound to use

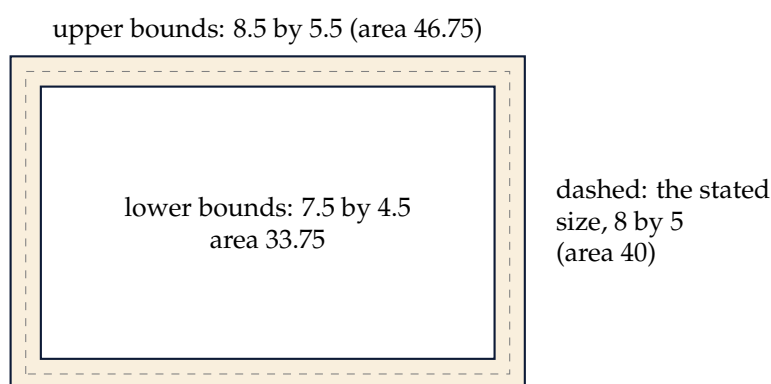
	greatest value	least value
$a + b$	$a_{\max} + b_{\max}$	$a_{\min} + b_{\min}$
$a - b$	$a_{\max} - b_{\min}$	$a_{\min} - b_{\max}$
$a \times b$	$a_{\max} \times b_{\max}$	$a_{\min} \times b_{\min}$
$a \div b$	$a_{\max} \div b_{\min}$	$a_{\min} \div b_{\max}$

(for positive quantities). Ask each time: “does making this input bigger make the answer bigger or smaller?”

		a (smallest 6.35, largest 6.45)	
		$a = 6.35$	$a = 6.45$
b	$b = 2.45$	3.9	4.0
	$b = 2.55$	3.8	3.9

largest difference: biggest a minus smallest b : 4.0
smallest difference: smallest a minus biggest b : 3.8

The table above checks the rule for $a - b$ with $a = 6.4$ and $b = 2.5$, both to 1 d.p.: of the four combinations, the greatest difference is $6.45 - 2.45 = 4.0$ and the least is $6.35 - 2.55 = 3.8$. The rectangle below shows the same idea for an area: the product of the upper bounds is the outer rectangle, and the product of the lower bounds is the inner one.



Worked example: area

A rectangle is 8 cm by 5 cm, each to the nearest cm. Find the greatest and least possible area.
The sides lie in $7.5 \leq L < 8.5$ and $4.5 \leq W < 5.5$. Greatest area = $8.5 \times 5.5 = 46.75 \text{ cm}^2$; least area = $7.5 \times 4.5 = 33.75 \text{ cm}^2$.

Worked example: a quotient

$x = 6$ to the nearest integer and $y = 0.5$ to 1 d.p. Find the least possible value of $\frac{x}{y}$.
 $5.5 \leq x < 6.5$ and $0.45 \leq y < 0.55$. The least quotient is the smallest numerator over the largest denominator:
 $\frac{5.5}{0.55} = 10$.

Worked example: speed

A runner covers 100 m (to the nearest metre) in 12.5 s (to the nearest 0.1 s). Find the greatest possible average speed.
Speed = $\frac{\text{distance}}{\text{time}}$, so the greatest speed uses the greatest distance and the least time: $\frac{100.5}{12.45} \approx 8.07 \text{ m/s}$. (Using 100 and 12.5 would give exactly 8, which is not the greatest.)

Common mistakes

- Using the lower bound of b in $a - b$ for the least value. Subtraction reverses the role of b .
- Using the lower bound of the denominator to find the *least* quotient. The least quotient divides by the *largest* denominator.
- Adding the errors incorrectly: for a perimeter $2(L + W)$ the upper bound uses the upper bounds of L and W (a sum, so no reversing).
- Quoting a bound as if it were the answer to a calculation of the unrounded numbers: the question may ask for the bound, not the stated value.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Evaluate $25^{-3/2}$.
2. Solve $9^x = 27$.
3. Simplify $\frac{2^{n+2} - 2^n}{2^{n-1}}$.
4. Simplify $\sqrt{50} + \sqrt{8}$.
5. Expand and simplify $(\sqrt{7} + 2)(\sqrt{7} - 2)$.
6. Rationalise $\frac{4}{3 - \sqrt{5}}$.
7. Write the error interval for $x = 340$ to the nearest 10, and the greatest possible value of $2x$ (as an upper bound).
8. A rectangle has sides 8 cm and 5 cm, each to the nearest cm. Find the upper bound of its perimeter.

Answers to the self-check

1. $25^{3/2} = (\sqrt{25})^3 = 125$, so $25^{-3/2} = \frac{1}{125}$.
2. $3^{2x} = 3^3$, so $2x = 3$ and $x = \frac{3}{2}$.
3. Factor 2^n : $\frac{2^n(4-1)}{2^n \times 2^{-1}} = 3 \times 2 = 6$.
4. $5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}$.
5. $7 - 4 = 3$.
6. $\frac{4(3 + \sqrt{5})}{9 - 5} = 3 + \sqrt{5}$.
7. $335 \leq x < 345$; the upper bound of $2x$ is 690.
8. Upper bounds 8.5 and 5.5, so $2(8.5 + 5.5) = 28$ cm.

Next step

Heurisco's *ESAT Maths 1: Indices, Surds and Bounds Practice Set* has twenty original exam-style questions on this lesson, with diagrams, in order of difficulty, with a full worked solution for each.