

ESAT

Maths 1

Ratio, proportion and percentages

Free lesson notes

Covers	M3.1 to M3.11
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Ratio and proportion questions look wordy but almost always reduce to one idea: find the value of *one part* (one unit, one percent, one worker for one day) and then scale. Percentage questions reduce to a *multiplier*, and learning to think in multipliers makes reverse percentages, repeated change and compound interest all the same calculation. Applied questions (mixtures, work rates, speeds, maps) are these same ideas dressed up, and the picture, usually a bar or a double number line, does most of the work.

This lesson builds the topic from first principles. We begin with what a ratio is and share quantities with bar models; combine ratios and handle fractions inside ratio problems; use the unitary method and double number lines for proportion, scale and unit conversion; meet percentages and multipliers; extend to simple and compound interest, growth and decay; study direct and inverse proportion with their graphs (including work-rate problems); look at similar shapes, where lengths, areas and volumes scale differently; and finish with mixtures and other applied reasoning.

How to use it. Read each idea, then cover the worked example and try it yourself. Draw the bar or number line every time: it is faster than it looks. Finish with the self-check, then the practice set. Specification codes M3.1 to M3.11.

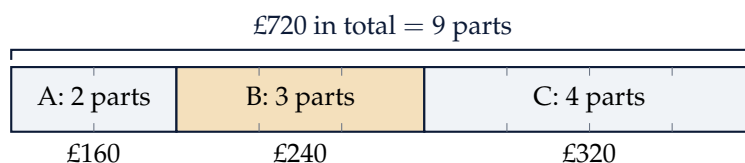
Heurisco is an independent practice provider and is not affiliated with, endorsed by, or connected to UAT-UK, Pearson VUE, the UCAT Consortium, or any university. All questions are original.

1 Ratio from scratch and bar models

What a ratio says. A ratio compares quantities of the same kind. "Squash and water in the ratio 1 : 4" means that for every 1 cup of squash there are 4 cups of water. Nothing is said about the total amount: 1 cup and 4 cups, or 10 and 40, all have ratio 1 : 4. As with fractions we may multiply or divide every part by the same number: $12 : 18 = 2 : 3$ (divide by 6). A ratio is in its *simplest form* when the parts have no common factor. To remove decimals or fractions first multiply through: $0.5 : 2 = 1 : 4$ and $\frac{1}{2} : \frac{3}{4} = 2 : 3$ (multiply by 4).

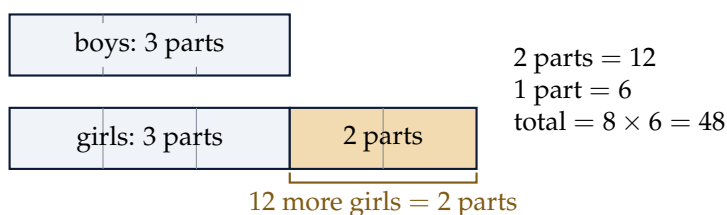
Part-to-part versus part-to-whole. In the ratio 1 : 4 there are $1 + 4 = 5$ equal parts altogether. So the squash is $\frac{1}{5}$ of the mixture, not $\frac{1}{4}$. Whenever you meet a ratio, *add the parts* to find how many equal slices make the whole.

Sharing in a ratio with a bar model. To share £720 in the ratio 2 : 3 : 4, draw a bar cut into equal parts: $2 + 3 + 4 = 9$ parts. The whole bar is £720, so each part is $720 \div 9 = £80$, and the shares are 2×80 , 3×80 and 4×80 . The recipe is always: *add the parts, find one part, multiply.*



$$1 \text{ part} = 720 \div 9 = £80, \text{ so the shares are } 2 \times 80, 3 \times 80, 4 \times 80$$

Using a difference. Sometimes we are told how much bigger one part is, not the total. The ratio of boys to girls is 3 : 5 and there are 12 more girls than boys. Draw both bars. The girls' bar is longer by $5 - 3 = 2$ parts, and that excess is 12 people, so one part is 6 and the whole group is 8 parts = 48. The bar shows which parts correspond to the given number.



Worked example: sharing

Ann, Ben and Cat share £840 in the ratio 3 : 4 : 5. How much more than Ann does Cat receive?
Total parts = 12, so one part = $840 \div 12 = £70$. Ann gets $3 \times 70 = 210$ and Cat $5 \times 70 = 350$. The difference is £140 (or directly: 2 parts = 2×70).

Worked example: ratio in context

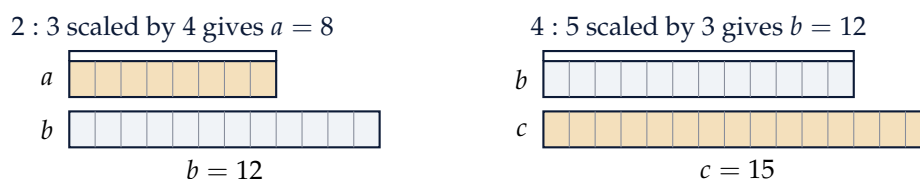
The angles of a triangle are in the ratio 2 : 3 : 4. Find the largest angle.
The angles sum to 180° and there are 9 parts, so one part is 20° and the largest angle is $4 \times 20 = 80^\circ$.

Common mistakes

- Dividing the quantity by the *number of terms* rather than the *sum of the parts*. For 3 : 4 : 5 divide by $3 + 4 + 5 = 12$, not by 3.
- Reading $a : b$ as the fraction $\frac{a}{b}$ of the whole. In 3 : 5 the boys are $\frac{3}{8}$ of the total, not $\frac{3}{5}$.
- Using the difference of the *quantities* as one part. A difference of 12 stands for 2 parts, not 1.

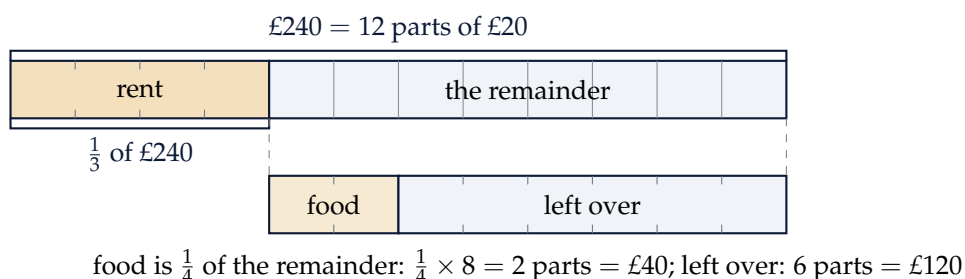
2 Combining ratios and fractions in ratio problems

Linking two ratios through a common term. If $a : b = 2 : 3$ and $b : c = 4 : 5$, the two ratios both mention b , but with different values (3 in one, 4 in the other). Multiply each ratio so that b is the same in both: scale $a : b$ by 4 to get $8 : 12$ and scale $b : c$ by 3 to get $12 : 15$. Now $b = 12$ in both, so $a : b : c = 8 : 12 : 15$, and in particular $a : c = 8 : 15$. The picture shows the bars lining up.



so $a : b : c = 8 : 12 : 15$ and $a : c = 8 : 15$

Fractions of a quantity and “of what is left”. A fraction like $\frac{1}{3}$ of £240 is one of three equal parts: $240 \div 3 = £80$. When a problem says “a quarter of the rest”, the quarter is of the remainder, not of the original total. Draw the whole bar, mark off the first fraction, and then split what is left. In the bar below, rent is $\frac{1}{3}$ of £240 (that is 4 of 12 parts); the remainder is 8 parts, and food is a quarter of that, 2 parts, = £40.



Changing a ratio when something is added or removed. The quantity that does *not* change is the anchor: fix it, express the other part in the new ratio, and subtract.

Worked example: adding water

A drink is made from concentrate and water in the ratio 1 : 4. How much water must be added to 600 ml of the drink to make the ratio 1 : 6?
The concentrate is $\frac{1}{5}$ of $600 = 120$ ml and stays fixed; the water is 480 ml. For the ratio 1 : 6 the water must be $6 \times 120 = 720$ ml. So add $720 - 480 = 240$ ml.

Worked example: fractions of what is left

Ann receives $\frac{3}{5}$ of a sum, Ben receives $\frac{1}{4}$ of what is left, and Cat gets the rest. Cat gets £90 more than Ben. Find the sum.

After Ann, $\frac{2}{5}$ remains. Ben gets $\frac{1}{4} \times \frac{2}{5} = \frac{1}{10}$ of the whole and Cat $\frac{3}{4} \times \frac{2}{5} = \frac{3}{10}$. Cat has $\frac{2}{10} = \frac{1}{5}$ more than Ben, so $\frac{1}{5}$ of the sum is £90 and the sum is £450.

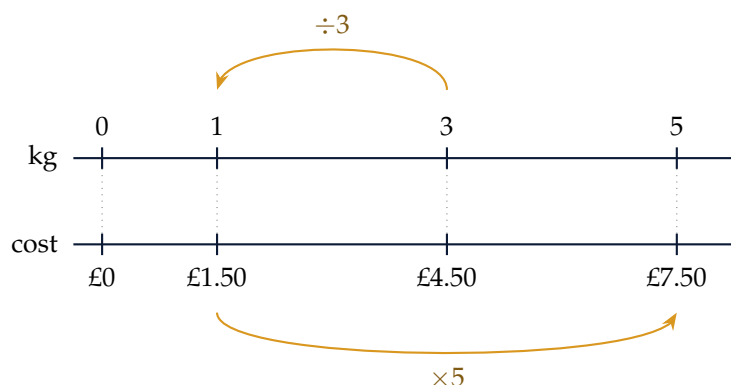
Common mistakes

- Cancelling the common letter without matching its value first: $a : b = 2 : 3$ and $b : c = 4 : 5$ does *not* give $a : c = 2 : 5$.
- Taking “a quarter of what is left” as a quarter of the original amount.
- Changing the quantity that stays fixed. In a mixing problem, only the thing you add changes.

3 Proportion, the unitary method and scale

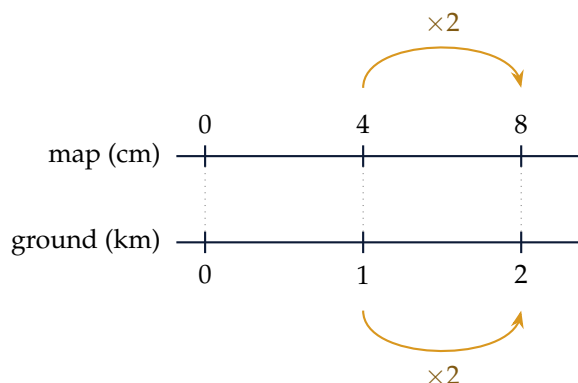
Two quantities are in *proportion* when one is a constant multiple of the other: doubling one doubles the other. The *unitary method* finds the value of *one unit* and then scales.

The double number line. Put one quantity on the top line and the other underneath, with matching marks. If 3 kg costs £4.50, divide both by 3 to get 1 kg for £1.50, then multiply both by 5 to get 5 kg for £7.50. Whatever you do to the top you do to the bottom.



Best buys. To compare deals, find the price of one unit of each: the cheaper price per unit is the better value. For example 500 g for £1.20 is £2.40 per kg, while 750 g for £1.65 is £2.20 per kg, which is better.

Scale and maps. A map scale 1 : 25 000 is a ratio of lengths: 1 cm on the map is 25 000 cm on the ground. Convert at the end: 25 000 cm = 250 m, so 4 cm on the map is 1 km on the ground, and 8 cm is 2 km.



Worked example: a recipe

A recipe for 6 people uses 450 g of flour and 3 eggs. How much of each is needed for 10 people?
For 1 person: 75 g and $\frac{1}{2}$ egg. For 10 people: 750 g and 5 eggs. (Multiplier: $\frac{10}{6}$.)

Worked example: units and cost

A car uses 6 litres of petrol per 100 km. Petrol costs £1.50 per litre. What is the cost of a 250 km journey?
250 km is 2.5 lots of 100 km, so the car uses $2.5 \times 6 = 15$ litres. Cost = $15 \times 1.50 = £22.50$.

Worked example: best buy

Which is better value: 750 g of rice for £1.80 or 2 kg for £4.60?
Per kg: $1.80 \div 0.75 = £2.40$ and $4.60 \div 2 = £2.30$. The 2 kg bag is slightly better value.

Common mistakes

- Mixing units: 1 km is 1000 m, but a map scale in cm needs dividing by 100 and then by 1000.
- Comparing total prices of different quantities. Always compare the price per unit.
- Scaling only one line of a double number line.

4 Percentages and multipliers

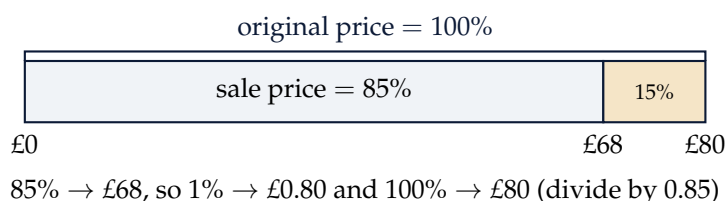
Percent means "out of 100". $35\% = \frac{35}{100} = 0.35$. To find 35% of a number, multiply by 0.35. Quick routes: 10% is $\div 10$, 1% is $\div 100$, so $35\% = 3 \times 10\% + 5\%$ where 5% is half of 10%.

Percentage change as a multiplier. To increase a quantity by 20% you keep the whole 100% and add 20% more, so you have $120\% = 1.2$ times as much. To decrease by 15% you keep $85\% = 0.85$ of it. So

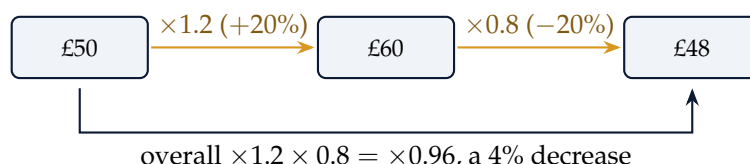
$$\text{increase by } p\% \rightarrow \times \left(1 + \frac{p}{100}\right) \quad \text{decrease by } p\% \rightarrow \times \left(1 - \frac{p}{100}\right).$$

A multiplier does the whole job in one step, and successive changes simply *multiply* their multipliers.

Reverse percentages. If a price after a 15% discount is £68, then £68 is 85% of the original. The bar shows it: $85\% \rightarrow £68$, so $1\% \rightarrow £0.80$ and $100\% \rightarrow £80$. Equivalently, *divide by the multiplier*: $68 \div 0.85 = 80$. Never increase the sale price by 15%: that finds 15% of the *sale* price, which is a different amount.



Up then down. A price of £50 rises by 20% and then falls by 20%. The multipliers are 1.2 and 0.8, so the final price is $50 \times 1.2 \times 0.8 = 50 \times 0.96 = £48$. The overall change is a 4% *decrease*: the fall is 20% of the larger number.



Percentage of a percentage; percentage points. Taking 25% of 40% gives $0.25 \times 0.4 = 0.10 = 10\%$. Be careful with the words: if a rate rises from 20% to 25%, it has risen by 5 *percentage points*, but by $5 \div 20 = 25\%$ *in percentage terms*.

Finding a percentage change. $\text{change} = \frac{\text{new} - \text{old}}{\text{old}} \times 100\%$. The *original* value is always the denominator.

Worked example: area after a change in lengths

A rectangle's length is increased by 20% and its width decreased by 10%. What is the percentage change in area? The multipliers are 1.2 and 0.9, so area is multiplied by $1.2 \times 0.9 = 1.08$: an 8% increase.

Worked example: weighted percentages

In a class, 40% are girls. 25% of the girls and 50% of the boys wear glasses. What percentage of the class wears glasses?

Take 100 students: 40 girls and 60 boys. Glasses: $0.25 \times 40 + 0.5 \times 60 = 10 + 30 = 40$. So 40%. (Not 37.5%: the groups are different sizes.)

Worked example: percentage change

A train fare rises from £48 to £54. What is the percentage increase?

Change = 6; original = 48. $\frac{6}{48} = 0.125 = 12.5\%$.

Common mistakes

- Finding the original by *increasing* the sale price by 15%. Divide by 0.85 instead.
- Thinking +20% then -20% returns to the start. The decrease is 20% of a larger amount.
- Using the *new* value as the denominator for percentage change.
- Adding percentage changes of dimensions to get the change in area: multiply the multipliers.

5 Interest, growth and decay

Simple interest is the same amount every year: £2000 at 3% earns $0.03 \times 2000 = £60$ a year, so after 4 years the total is $2000 + 4 \times 60 = £2240$. The value grows by a fixed amount: a straight line.

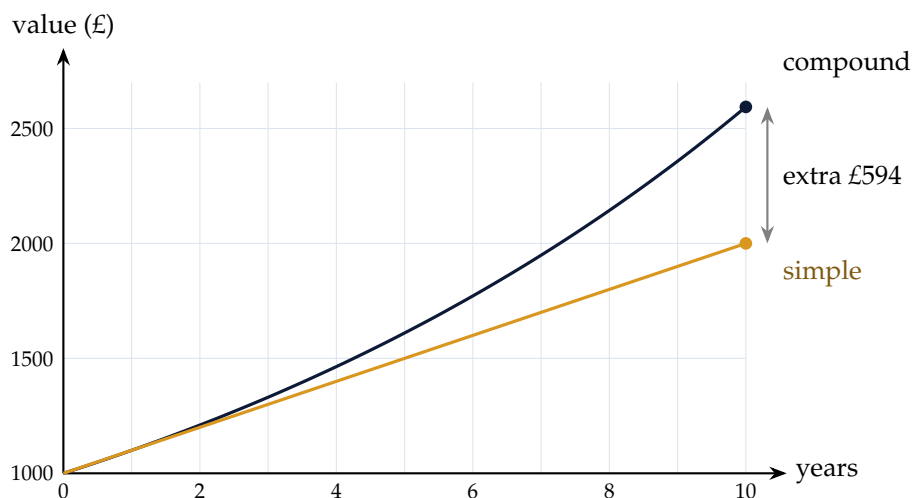
Compound interest. After year 1 the interest has joined the balance, so next year interest is earned on the larger amount (interest on interest). In terms of multipliers: after one year the balance is $\times 1.04$; after the second year it is multiplied by 1.04 *again*:

$$£5000 \rightarrow 5000 \times 1.04 \rightarrow 5000 \times 1.04 \times 1.04 = 5000 \times 1.04^2 = £5408.$$

In general

$$\text{final} = \text{start} \times m^n$$

where m is the multiplier for one period and n is the number of *periods*. The graph compares £1000 at 10% simple (a line) with 10% compound (a curve). The curve bends upwards because each year's increase is bigger than the last.



Decay. Compound decay uses a multiplier below 1. A van worth £20 000 loses 10% of its value each year, so its value is multiplied by 0.9 each year. Notice that each year's loss is 10% of the *current* value, so the losses get smaller: 2000, 1800, 1620. After 3 years: $20\,000 \times 0.9^3 = £14\,580$, not £14 000.

each year loses 10% of the *current* value: 2000, then 1800, then 1620



Doubling and other periods. If the quantity doubles every 3 hours, the multiplier is 2 per *three-hour period*. After 12 hours there are $12 \div 3 = 4$ periods, so the amount is multiplied by $2^4 = 16$. Always count *periods* (not years or hours) when the rate applies per period.

Worked example: compound interest

£5000 earns 4% compound interest a year. What is the value after 2 years?

$5000 \times 1.04^2 = 5000 \times 1.0816 = £5408$. (Simple interest would give £5400: the extra £8 is interest on the first year's interest.)

Worked example: depreciation

A van worth £20 000 loses 10% of its value each year. What is it worth after 3 years?

$20\,000 \times 0.9^3 = 20\,000 \times 0.729 = £14\,580$.

Worked example: doubling

There are 500 bacteria and the number doubles every 3 hours. How many are there after 15 hours?

$15 \div 3 = 5$ doublings. $500 \times 2^5 = 500 \times 32 = 16\,000$.

Common mistakes

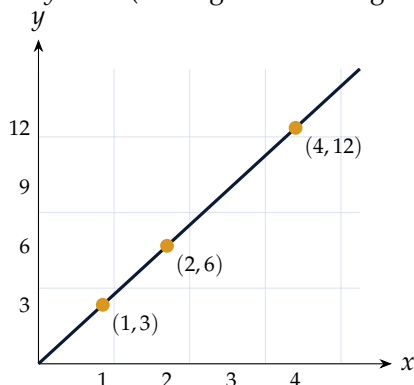
- Using $5000 \times 1.04 \times 2$: powers, not multiples, give compound change.
- Computing 0.9^3 as 0.27. $0.9^3 = 0.9 \times 0.9 \times 0.9 = 0.729$.
- Counting years when the interest period is a month or a half-year: n is the number of *periods*.
- Treating exponential growth as linear when extrapolating from a graph.

6 Direct and inverse proportion

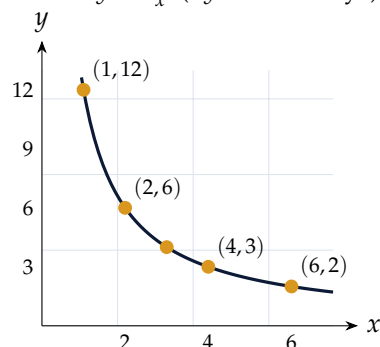
Direct proportion. y is directly proportional to x , written $y \propto x$, when y is always the same multiple of x : $y = kx$. The graph is a straight line through the origin, and doubling x doubles y . The number k is the constant of proportionality; find it from any given pair.

Inverse proportion. y is inversely proportional to x , $y \propto \frac{1}{x}$, when $y = \frac{k}{x}$, or equivalently $xy = k$ is constant. Doubling x halves y . The graph is a curve that falls steeply and flattens, never touching either axis. Both graphs below pass through (2, 6), but one rises and one falls.

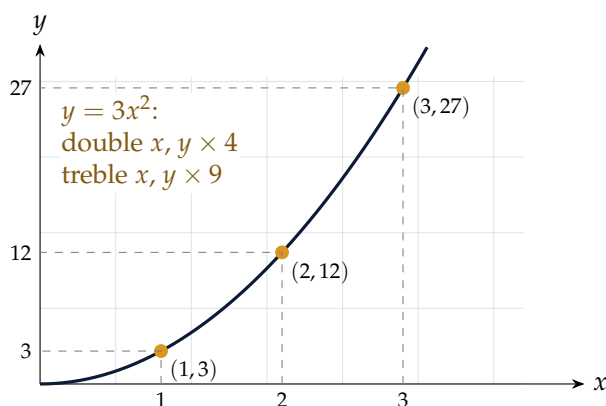
direct: $y = 3x$ (a straight line through O)



inverse: $y = \frac{12}{x}$ ($xy = 12$ always)

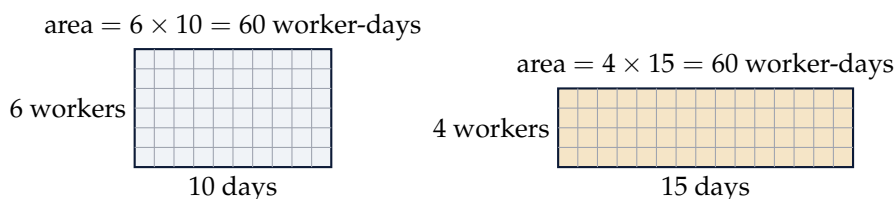


Other powers. $y \propto x^2$ means $y = kx^2$. Doubling x gives $(2x)^2 = 4x^2$, so y is multiplied by 4; trebling x multiplies y by 9. The graph is a parabola through the origin.



A recipe for every problem: (1) write the relationship with k , (2) use the given pair to find k , (3) use k to answer.

Work rates are inverse proportion. Suppose 6 workers take 10 days. The *total work* is $6 \times 10 = 60$ worker-days: the area of a rectangle with sides workers and days. Another team with different numbers must cover the *same area*: 4 workers need $60 \div 4 = 15$ days. The picture shows two rectangles with equal area.



Worked example: direct, squared

$y \propto x^2$ and $y = 12$ when $x = 2$. Find y when $x = 5$.
 $12 = k \times 2^2$ so $k = 3$. Then $y = 3 \times 25 = 75$.

Worked example: inverse

y is inversely proportional to x and $y = 6$ when $x = 4$. Find x when $y = 8$.
 $xy = k = 4 \times 6 = 24$. When $y = 8$, $x = 24 \div 8 = 3$. (As y increases, x decreases.)

Worked example: a change in percentage terms

y is directly proportional to the square of x . By what percentage does y increase when x increases by 50%?
 $x \rightarrow 1.5x$ so $y \rightarrow 1.5^2 y = 2.25y$. That is an increase of 125%.

Worked example: work done in stages

5 identical pumps empty a tank in 12 hours. After 3 hours, 4 more identical pumps are switched on. How many hours does the whole job take?

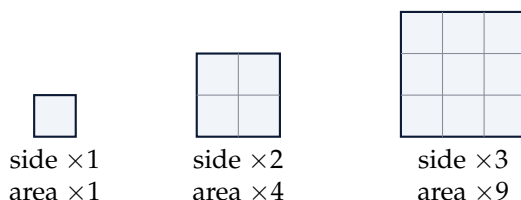
Tank = $5 \times 12 = 60$ pump-hours. In 3 hours 5 pumps do 15, so 45 remain. With 9 pumps that takes $45 \div 9 = 5$ hours. Total: $3 + 5 = 8$ hours.

Common mistakes

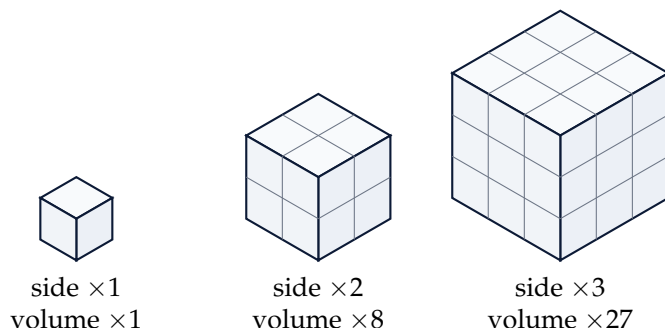
- Treating an inverse relationship as direct: more workers means *fewer* days.
- For $y \propto x^2$, scaling x by 3 scales y by 9, not 3.
- Using a graph that does not pass through the origin as direct proportion. Direct proportion always goes through $(0, 0)$.

7 Similar shapes: lengths, areas and volumes

Two shapes are *similar* if one is an enlargement of the other: all lengths are multiplied by the same scale factor k . But areas and volumes do *not* scale by k . A square of side 1 has area 1. Double the side and the square needs $2 \times 2 = 4$ unit squares; treble it and it needs $3 \times 3 = 9$. Areas scale by k^2 .



In three dimensions a cube of side 2 is made of $2 \times 2 \times 2 = 8$ unit cubes, and a cube of side 3 of 27. Volumes scale by k^3 .



Scale factors

lengths $\times k$ areas $\times k^2$ volumes (and masses, for the same material) $\times k^3$

To go from an area ratio or volume ratio back to a length ratio, take the square root or the cube root.

Worked example: volume ratio

Two similar cones have heights 6 cm and 9 cm. The smaller has volume 80 cm^3 . Find the volume of the larger. Length scale factor $\frac{9}{6} = \frac{3}{2}$, so volume scale factor $\left(\frac{3}{2}\right)^3 = \frac{27}{8}$. Volume = $80 \times \frac{27}{8} = 270 \text{ cm}^3$.

Worked example: from areas to volumes

The surface areas of two similar solids are in the ratio $4 : 9$. The smaller has volume 32 cm^3 . Find the larger volume. Length ratio = $\sqrt{4} : \sqrt{9} = 2 : 3$, so volume ratio $2^3 : 3^3 = 8 : 27$. Larger volume = $32 \times \frac{27}{8} = 108 \text{ cm}^3$.

Worked example: maps and area

On a map of scale 1 : 20 000 a field is a rectangle 2 cm by 1.5 cm. What is the real area in hectares? (1 hectare = 10 000 m².)

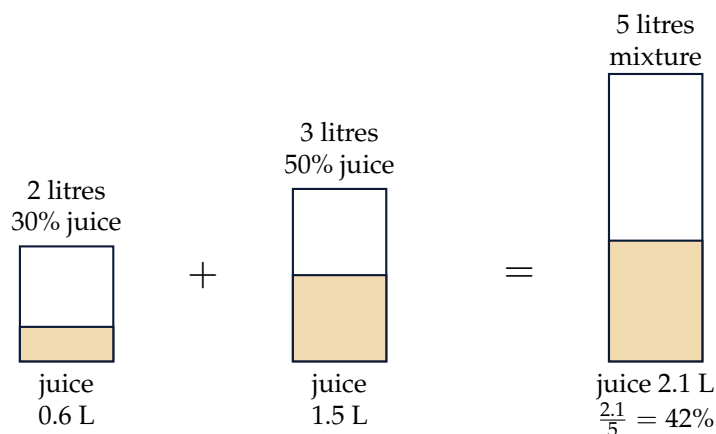
Map area = 3 cm². Real area = $3 \times 20\,000^2 \text{ cm}^2 = 1.2 \times 10^9 \text{ cm}^2 = 1.2 \times 10^5 \text{ m}^2$ (dividing by 100²), which is 12 hectares. Check by lengths: the field is 400 m by 300 m, and $400 \times 300 = 120\,000 \text{ m}^2$.

Common mistakes

- Using the length ratio for areas or volumes. A length ratio of 2 : 3 gives areas 4 : 9 and volumes 8 : 27.
- Cubing the *area* ratio, or squaring it when you want volume. Convert to the length ratio first.
- Squaring only the scale and forgetting the unit change: $1 \text{ m}^2 = 10\,000 \text{ cm}^2$, not 100.

8 Mixtures and applied reasoning

Mixtures are weighted averages. Mix 2 litres of juice that is 30% fruit with 3 litres that is 50% fruit. Add up the *pure fruit*: $0.3 \times 2 = 0.6 \text{ L}$ and $0.5 \times 3 = 1.5 \text{ L}$, total 2.1 L in 5 L, so the mixture is $\frac{2.1}{5} = 42\%$ fruit. It sits between 30% and 50%, nearer 50% because there is more of that juice.



Working backwards: the ratio needed. To get a mixture of strength 42% from 30% and 50% juice, notice that 42 is 12 above 30 and 8 below 50. The amounts must balance like a seesaw: more of the juice that is *closer* to the target. The ratio $A : B = (50 - 42) : (42 - 30) = 8 : 12 = 2 : 3$. Check with algebra: $\frac{30 \times 2 + 50 \times 3}{5} = \frac{210}{5} = 42$.

Speed, distance and time with ratios. If speed is constant, distance and time are in direct proportion; if the distance is fixed, speed and time are in inverse proportion. A journey at 60 km/h taking 2 h takes $\frac{60}{80} \times 2 = 1.5 \text{ h}$ at 80 km/h: speed up by $\frac{4}{3}$, time down by $\frac{3}{4}$. Convert units *before* comparing: 90 km/h is 25 m/s.

Estimating and sanity-checking. Percentage answers should make sense: a discount of 15% cannot give a price *higher* than the original, and a mixture cannot be stronger than its strongest ingredient. Use these facts to eliminate options quickly.

Worked example: mixing

In what ratio must a 30% juice and a 50% juice be mixed to give a 42% juice?

Let the amounts be a and b . $\frac{30a + 50b}{a + b} = 42$, so $30a + 50b = 42a + 42b$, giving $8b = 12a$ and $a : b = 2 : 3$.

Worked example: an unknown percentage

A price is increased by $x\%$ and then the new price is decreased by $x\%$. The final price is 1% lower than the original. Find x .

The multiplier is $(1 + \frac{x}{100})(1 - \frac{x}{100}) = 1 - \frac{x^2}{10\,000}$ (difference of two squares). Setting this equal to 0.99 gives $x^2 = 100$ and $x = 10$.

Worked example: speed and ratio

A cyclist rides 12 km at 18 km/h and then a further 12 km at 9 km/h. What is the average speed for the whole journey?

Times: $\frac{12}{18} = \frac{2}{3}$ h and $\frac{12}{9} = \frac{4}{3}$ h, total 2 h. Distance 24 km. Average = $24 \div 2 = 12$ km/h (not 13.5, the average of the speeds).

Exam tip

Draw a bar or double number line before calculating. Check each answer against common sense: is a percentage change positive or negative? Is the answer bigger or smaller than the starting value?

Common mistakes

- Averaging percentages or speeds instead of weighting by the amounts.
- Mixing up which ingredient has the larger share in a mixture ratio: the ratio is between the *amounts*, and the mixture is closer to the larger amount.
- Forgetting to convert units before comparing speeds or rates.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Share £720 in the ratio 2 : 3 : 4. What is the largest share?
2. $a : b = 3 : 4$ and $b : c = 6 : 5$. Find $a : c$.
3. A jacket costs £91 after a 30% discount. What was the original price?
4. A price rises by 5% then falls by 10%. What single multiplier does this equal?
5. £800 earns 5% compound interest for 2 years. Find the total.
6. y is inversely proportional to x . $y = 4$ when $x = 9$. Find y when $x = 6$.
7. Two similar shapes have areas 50 cm^2 and 200 cm^2 . What is the ratio of their lengths, and of their volumes (if solid)?
8. How many litres of 20% solution must be added to 3 litres of 40% solution to make a 30% solution?

Answers to the self-check

1. Total parts 9; $720 \div 9 = 80$; largest $= 4 \times 80 = £320$.
2. $a : b = 9 : 12$ and $b : c = 12 : 10$, so $a : c = 9 : 10$.
3. $91 \div 0.7 = £130$.
4. $1.05 \times 0.9 = 0.945$ (an overall decrease of 5.5%).
5. $800 \times 1.05^2 = 800 \times 1.1025 = £882$.
6. $xy = 36$, so $y = 36 \div 6 = 6$.
7. Area ratio 1 : 4, so length ratio 1 : 2 and volume ratio 1 : 8.
8. 30% is exactly midway between 20% and 40%, so the two amounts must be equal: 3 litres. (Check: $0.2 \times 3 + 0.4 \times 3 = 1.8 = 0.3 \times 6$.)

Next step

Heurisco's *ESAT Maths 1: Ratio, Proportion and Percentages Practice Set* has twenty original exam-style questions on this lesson, with diagrams, in order of difficulty, with a full worked solution for each.