

ESAT

Maths 1

Algebra

Free lesson notes

Covers

M4.1 to M4.8

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

Algebra is the language of the whole Maths 1 paper. Geometry, graphs, sequences and probability questions all turn into algebra once you start working on them, so speed and accuracy here pay off everywhere. It is also the part of mathematics where a few well-chosen pictures replace pages of symbol-pushing, and this lesson uses them heavily: areas for expanding and factorising, function machines for rearranging, and graphs for identities.

We build the subject in eight steps. First the notation and what a substitution really does; then the index laws, derived from scratch; then expanding brackets (with area models), factorising (expanding run backwards), algebraic fractions, rearranging formulae, equations versus identities including completing the square, and finally algebraic proof, where you show that something is true for *every* number rather than just for the examples you tried.

How to use it. Read each idea, then cover the worked example and try it yourself before reading the solution. Every worked example has been checked by computer, but do not trust a method until you have used it yourself. Finish with the self-check on the last pages. On the ESAT you have about 90 seconds a question and no calculator, so the aim is not only to get the right answer but to see the *short route* to it. Specification codes covered: M4.1 to M4.8.

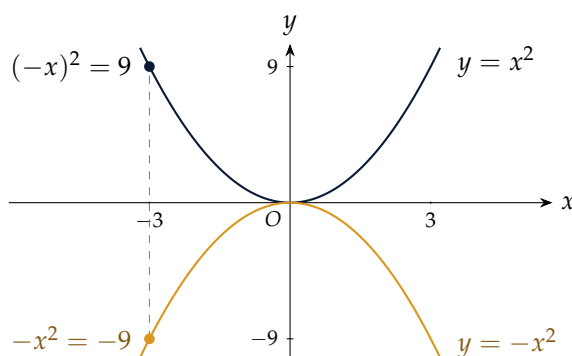
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1 Letters, notation and substitution

A *letter* in algebra stands for a number: either one we do not know yet, or one that is allowed to change. Writing the arithmetic with letters lets us describe a whole family of calculations at once. A few words are worth getting straight, because exam questions use them precisely.

- An **expression** is a collection of letters, numbers and operations with no equals sign: $3x^2 - 5x + 1$.
- An **equation** says two expressions are equal, and is usually true only for particular values: $3x - 5 = x + 1$ is true only for $x = 3$.
- A **formula** is an equation linking quantities, such as $A = \frac{1}{2}bh$.
- An **identity** is true for *every* value of the letters: $(x + 1)^2 \equiv x^2 + 2x + 1$. (Section 7.)
- A **term** is one chunk of an expression separated by $+$ or $-$. The **coefficient** of a term is the number multiplying the letters: in $-5x$ it is -5 .

Conventions. So that expressions are short and unambiguous, algebra drops the multiplication sign: $3x$ means $3 \times x$, xy means $x \times y$ and x^2 means $x \times x$. A fraction bar acts like a bracket: $\frac{x+1}{2}$ means $(x+1) \div 2$. A power applies *only to the thing it touches*: $2x^2 = 2 \times (x^2)$ but $(2x)^2 = 4x^2$. This is the most common source of lost marks in the whole paper, so look at the picture below. For $x = -3$ the expression $(-x)^2$ is $(+3)^2 = 9$ (a point on $y = x^2$), but $-x^2$ means “square first, then change the sign”, which is -9 (a point on the reflected curve $y = -x^2$).



Substitution means replacing each letter by a number. Always put *brackets* round the number you substitute, particularly if it is negative; then follow BIDMAS (brackets, indices, division and multiplication, addition and subtraction). The brackets stop a minus sign from slipping away.

Worked example: substitution with negatives

Find the value of $2a^2 - ab + b$ when $a = -2$ and $b = 3$.
 Substitute with brackets: $2(-2)^2 - (-2)(3) + 3$.
 Indices first: $(-2)^2 = 4$, so $2 \times 4 = 8$.
 Then $-(-2)(3) = -(-6) = +6$.
 Total: $8 + 6 + 3 = 17$.

Worked example: a fraction with negatives

Find the value of $\frac{3p - q}{p^2 + q}$ when $p = -2$ and $q = -5$.
 Numerator: $3(-2) - (-5) = -6 + 5 = -1$. Denominator: $(-2)^2 + (-5) = 4 - 5 = -1$.
 The fraction is $\frac{-1}{-1} = 1$.

Common mistakes

- Writing $-2^2 = 4$. The index applies to the 2 only, so $-2^2 = -4$; use $(-2)^2$ if the minus belongs to the number.
- Forgetting that subtracting a negative adds: $5 - (-3) = 8$.
- Reading $2x^2$ as $(2x)^2$. Powers apply only to what they touch.

2 Powers and the index laws

A power is repeated multiplication: $a^4 = a \times a \times a \times a$. The number 4 is the *index* (plural *indices*). The index laws are not arbitrary rules to memorise; each one can be seen by writing the product out.

Multiplying. $a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7$: we have simply counted the number of a 's, so the indices *add*: $a^m \times a^n = a^{m+n}$.

Dividing. $\frac{a^5}{a^2} = \frac{a \times a \times a \times a \times a}{a \times a} = a^3$: two of the a 's cancel, so the indices *subtract*: $a^m \div a^n = a^{m-n}$.

Power of a power. $(a^2)^3 = a^2 \times a^2 \times a^2 = a^6$: three lots of two, so indices *multiply*: $(a^m)^n = a^{mn}$. Similarly $(ab)^n = a^n b^n$, which is why $(2x)^3 = 8x^3$ and not $2x^3$.

Zero and negative indices. Look at the pattern $2^3 = 8$, $2^2 = 4$, $2^1 = 2$: each step down divides by 2. Continuing the pattern gives $2^0 = 1$, then $2^{-1} = \frac{1}{2}$, $2^{-2} = \frac{1}{4}$. In general

$$a^0 = 1 \quad a^{-n} = \frac{1}{a^n}.$$

A negative index does *not* make the number negative: it means "take the reciprocal". (The same law $a^m \div a^n = a^{m-n}$ confirms it: $a^2 \div a^5 = a^{-3}$, and cancelling directly gives $\frac{1}{a^3}$.)

The index laws

$$a^m \times a^n = a^{m+n} \quad a^m \div a^n = a^{m-n} \quad (a^m)^n = a^{mn} \quad a^0 = 1 \quad a^{-n} = \frac{1}{a^n}$$

They only combine powers of the *same base*: $a^2 \times b^3$ and $a^2 + a^3$ cannot be simplified.

When an expression has numbers and several letters, deal with the numbers first, then each letter separately.

Worked example: simplifying a product of powers

Simplify $\frac{(3x^2y)^3}{9x^4y}$.
 Cube everything in the bracket: $(3x^2y)^3 = 27x^6y^3$.
 Divide numbers: $27 \div 9 = 3$. Subtract indices of x : $6 - 4 = 2$. Of y : $3 - 1 = 2$.
 Result: $3x^2y^2$.

Worked example: negative indices

Write $(2x^{-2})^{-3}$ without negative indices.

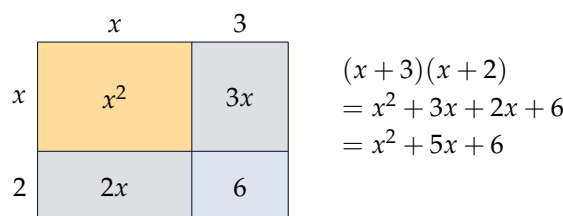
Apply the power to each factor: $2^{-3} \times x^{(-2)(-3)} = \frac{1}{8} \times x^6 = \frac{x^6}{8}$.

Common mistakes

- Adding the indices when dividing, or multiplying them when multiplying powers of the same base ($a^2 \times a^3 = a^5$, not a^6).
- Treating a^{-2} as $-a^2$ or $-2a$. It is $\frac{1}{a^2}$.
- Forgetting to apply the outside power to the number: $(3x^2)^3 = 27x^6$, not $3x^6$.

3 Expanding brackets

To *expand* a product of brackets, multiply every term in one bracket by every term in the other. An *area model* shows why this works. A rectangle with sides $(x + 3)$ and $(x + 2)$ has area equal to the sum of the areas of its four pieces.

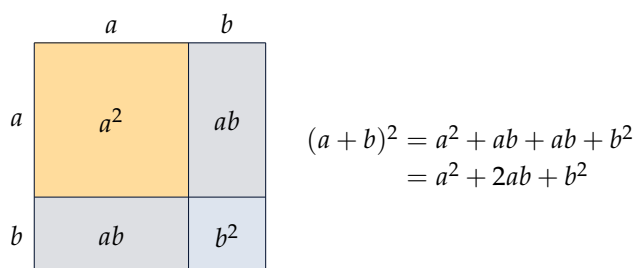


The four pieces are x^2 , $3x$, $2x$ and 6 , and the two middle pieces combine. In general

$$(x + p)(x + q) = x^2 + (p + q)x + pq :$$

the coefficient of x is the *sum* of the two numbers and the constant is their *product*. (Section 4 runs this backwards to factorise.) When the brackets contain minus signs, treat the minus as belonging to the number that follows it: $(x - 5)(x + 2)$ has numbers -5 and $+2$, giving $x^2 - 3x - 10$.

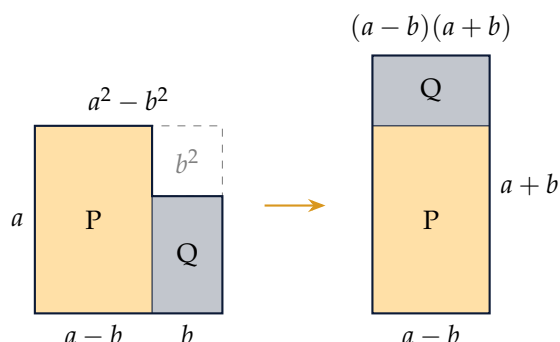
Squaring a bracket. Squaring means multiplying the bracket by itself, so it has *four* pieces, not two. The two middle pieces are equal.



$$(a + b)^2 = a^2 + 2ab + b^2 \quad (a - b)^2 = a^2 - 2ab + b^2.$$

The first term is the square of the first, the last term is the square of the last, and the middle term is twice the product. Writing $(a + b)^2 = a^2 + b^2$ is the single most common algebra error: the picture shows the two missing rectangles.

The difference of two squares. Multiplying $(a - b)(a + b)$ gives $a^2 + ab - ab - b^2 = a^2 - b^2$: the middle terms cancel. The picture shows the same fact: cut a small square $b \times b$ from the corner of the big square $a \times a$, then move the piece Q round to the top of piece P. The L-shape of area $a^2 - b^2$ becomes a rectangle of sides $(a - b)$ and $(a + b)$.



Three brackets. Expand two brackets first, then multiply the result by the third, taking care to multiply every term. A useful check is that the *constant term* of the answer is the product of the three numbers, and the coefficient of the second-highest power is their sum.

Cubes. Multiplying $(a+b)^2$ by $(a+b)$ gives $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Replace b by $-b$ to get $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$.

Worked example: expanding a perfect square and a product

Expand and simplify $(2x-3)^2 - (x+1)(x-4)$.

$(2x-3)^2 = 4x^2 - 12x + 9$ (first squared, twice the product $2 \times 2x \times (-3) = -12x$, last squared).

$(x+1)(x-4) = x^2 - 4x + x - 4 = x^2 - 3x - 4$.

Subtract (every term of the second bracket changes sign): $4x^2 - 12x + 9 - x^2 + 3x + 4 = 3x^2 - 9x + 13$.

Worked example: three brackets

Find the coefficient of x^2 in $(x+2)(x-1)(x+3)$.

First two brackets: $(x+2)(x-1) = x^2 + x - 2$.

Multiply by $(x+3)$: $x^3 + 3x^2 + x^2 + 3x - 2x - 6 = x^3 + 4x^2 + x - 6$.

The coefficient of x^2 is 4. (Checks: $4 = 2 + (-1) + 3$, the sum of the numbers, and the constant $-6 = 2 \times (-1) \times 3$.)

Worked example: using the difference of two squares for mental arithmetic

Evaluate $1001^2 - 999^2$ without a calculator.

Use $a^2 - b^2 = (a-b)(a+b)$ with $a = 1001$ and $b = 999$: $(1001-999)(1001+999) = 2 \times 2000 = 4000$.

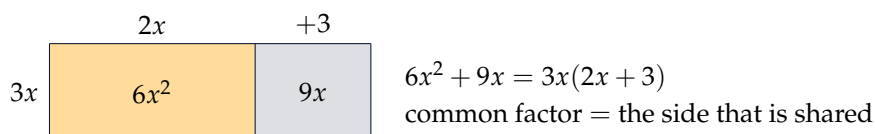
Common mistakes

- $(a+b)^2 \neq a^2 + b^2$. The middle term $2ab$ is always there.
- Sign slips in $(x-3)^2 = x^2 - 6x + 9$: the last term is positive because $(-3)^2 = +9$.
- Subtracting a product of brackets without changing the sign of every term: $-(x^2 - 3x - 4) = -x^2 + 3x + 4$.

4 Factorising

Factorising is expanding in reverse: we write an expression as a product. Work through these steps *in this order*.

Step 1: common factor. Take out the largest number and letters that divide every term. If a rectangle has area $6x^2 + 9x$ and one side is $3x$ (the common factor), the other side must be $2x + 3$.



Step 2: the difference of two squares. If there are two terms, both perfect squares with a minus between them, then $a^2 - b^2 = (a-b)(a+b)$ (Section 3). Remember that powers of letters are squares when the index is even: $49x^2 - 9y^4 = (7x)^2 - (3y^2)^2$.

Step 3: quadratics $x^2 + bx + c$. We saw that $(x + p)(x + q) = x^2 + (p + q)x + pq$. So to factorise $x^2 + bx + c$, find two numbers with *product* c and *sum* b . If c is positive the numbers have the same sign (the sign of b); if c is negative they have opposite signs, and the larger one carries the sign of b . For $x^2 - 5x - 14$ we need product -14 and sum -5 : the pair is -7 and $+2$, so $(x - 7)(x + 2)$.

Step 4: quadratics $ax^2 + bx + c$ with $a \neq 1$. Multiply $a \times c$ and find two numbers with that product and with sum b . These are the two middle pieces of the area model. Split the middle term into them and factorise in pairs. The grid below shows why it works for $6x^2 - x - 12$: the two pieces $+8x$ and $-9x$ have product $-72x^2 = 6x^2 \times (-12)$ and sum $-x$, and they fill the two off-diagonal cells.

	$3x$	$+4$
$2x$	$6x^2$	$+8x$
-3	$-9x$	-12

$6x^2 - x - 12$ has $ac = 6 \times (-12) = -72$.

The two middle terms must multiply to $-72x^2$ and add to $-x$: $+8x$ and $-9x$.

So $6x^2 - x - 12 = (3x + 4)(2x - 3)$.

Worked example: quadratic with $a \neq 1$

Factorise $6x^2 - x - 12$.

$ac = 6 \times (-12) = -72$. We need two numbers with product -72 and sum -1 : they are -9 and 8 .

Split the middle term: $6x^2 - 9x + 8x - 12$.

Factorise in pairs: $3x(2x - 3) + 4(2x - 3) = (3x + 4)(2x - 3)$.

Check by expanding: $6x^2 - 9x + 8x - 12 = 6x^2 - x - 12$. ✓

Worked example: common factor, then difference of squares

Factorise $18x^3 - 8x$ completely.

Common factor $2x$: $2x(9x^2 - 4)$. Now $9x^2 - 4 = (3x)^2 - 2^2$, so the result is $2x(3x - 2)(3x + 2)$.

Worked example: factorising by grouping

Factorise $x^3 + 2x^2 - 9x - 18$.

Group in pairs: $x^2(x + 2) - 9(x + 2) = (x + 2)(x^2 - 9) = (x + 2)(x - 3)(x + 3)$.

Exam tip

Always check a factorisation by expanding it mentally, looking at the first term, the last term and the middle term. It takes five seconds and catches nearly every error. If the check fails only on the middle term, you have the right numbers in the wrong places or with the wrong signs.

5 Algebraic fractions

Algebraic fractions obey exactly the same rules as numerical fractions. The one new danger is that you can only cancel *factors*, which are things multiplied together, never *terms*, which are things added.

To see why: $\frac{2 \times 3}{2 \times 5} = \frac{3}{5}$ is fine, because the 2 multiplies in both places. But $\frac{2+3}{2+5} = \frac{5}{7}$ is not $\frac{3}{5}$. In the same way $\frac{x+3}{x}$ cannot be simplified, but $\frac{3x}{x} = 3$ can.

- **Simplify:** factorise top and bottom completely, then cancel common factors.
- **Multiply:** multiply tops and bottoms (cancel factors first to keep numbers small).
- **Divide:** flip the second fraction and multiply. (Dividing by a fraction asks “how many of these fit?”, which is multiplying by its reciprocal.)
- **Add or subtract:** write both over a common denominator (the product of the denominators is always safe), then combine the numerators. Where a minus sign sits in front of a fraction it applies to the *whole* numerator, so use brackets.

A fraction is undefined where its denominator is zero, so $\frac{x-3}{x-2}$ has no value at $x = 2$. Cancelling a factor does not remove that restriction, but at ESAT level you are usually asked only to simplify.

Worked example: simplifying

Simplify $\frac{x^2 - 9}{x^2 + x - 6}$.

Numerator: $x^2 - 9 = (x - 3)(x + 3)$. Denominator: $x^2 + x - 6 = (x + 3)(x - 2)$ (product -6 , sum 1 : 3 and -2).

Cancel the common factor $(x + 3)$: $\frac{x - 3}{x - 2}$.

Worked example: subtracting fractions

Write $\frac{2}{x - 1} - \frac{3}{x + 2}$ as a single fraction.

Common denominator $(x - 1)(x + 2)$. Multiply the numerators by what is missing:

$$\frac{2(x + 2) - 3(x - 1)}{(x - 1)(x + 2)} = \frac{2x + 4 - 3x + 3}{(x - 1)(x + 2)} = \frac{7 - x}{(x - 1)(x + 2)}.$$

Worked example: dividing a fraction inside a fraction

Simplify $\frac{\frac{1}{x} - \frac{1}{x + 1}}{\frac{1}{x}}$.

Top: $\frac{1}{x} - \frac{1}{x + 1} = \frac{(x + 1) - x}{x(x + 1)} = \frac{1}{x(x + 1)}$.

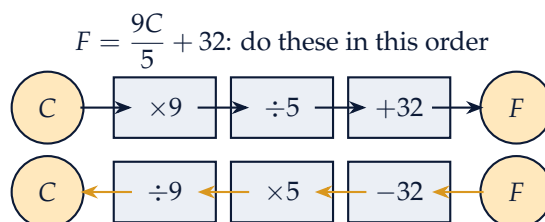
Dividing by $\frac{1}{x}$ is multiplying by x : $\frac{1}{x(x + 1)} \times x = \frac{1}{x + 1}$.

Common mistakes

- Cancelling terms instead of factors: in $\frac{x + 3}{x}$ you cannot cancel the x 's.
- When subtracting a fraction, the minus applies to the *whole* numerator: write $-3(x - 1)$, not $-3x - 1$.
- Adding numerators and denominators: $\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$. You need a common denominator.
- Forgetting to flip the second fraction when dividing.

6 Rearranging formulae

To make a letter the *subject* of a formula, we undo what has been done to it. Think of the formula as a function machine: the letter we want goes in, a sequence of operations is applied, and the result comes out. To get the letter back we run the machine *backwards*: start from the output, and undo the steps in the reverse order using the inverse operations (add $\leftrightarrow$ subtract, multiply $\leftrightarrow$ divide, square $\leftrightarrow$ square root).



undo the steps in the *reverse* order with the opposite operations: $C = \frac{5(F - 32)}{9}$

Doing the *same* to both sides of the equation is the same idea written as algebra: it keeps the equation balanced. After $F = \frac{9C}{5} + 32$: subtract 32, multiply by 5, divide by 9.

When the letter appears more than once you cannot undo it step by step. Instead, (1) clear any fractions by multiplying both sides, (2) expand brackets, (3) get every term containing the letter on one side and every other term on the other, (4) factorise the letter out, (5) divide.

Worked example: subject appears twice

Make x the subject of $y = \frac{2x+1}{x-3}$.
 Multiply both sides by $(x-3)$: $y(x-3) = 2x+1$.
 Expand: $xy - 3y = 2x + 1$.
 Collect the x terms on one side: $xy - 2x = 3y + 1$.
 Factorise: $x(y-2) = 3y+1$.
 Divide: $x = \frac{3y+1}{y-2}$.

Worked example: reciprocals

The lens formula is $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$. Make v the subject.
 $\frac{1}{v} = \frac{1}{f} - \frac{1}{u} = \frac{u-f}{uf}$. Taking the reciprocal of both sides: $v = \frac{uf}{u-f}$.

Worked example: rearrange, then substitute

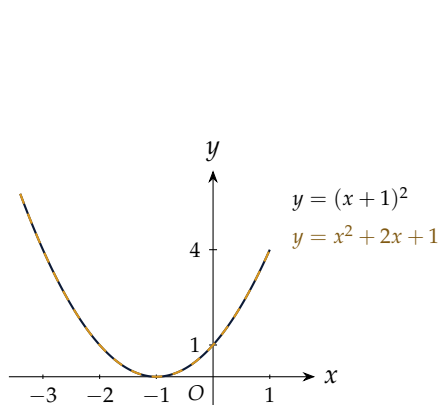
The formula $v^2 = u^2 + 2as$ is used with $v = 13$, $u = 5$ and $s = 6$. Find a .
 Rearrange first: $a = \frac{v^2 - u^2}{2s}$. Substitute: $a = \frac{169 - 25}{12} = \frac{144}{12} = 12$.

Common mistakes

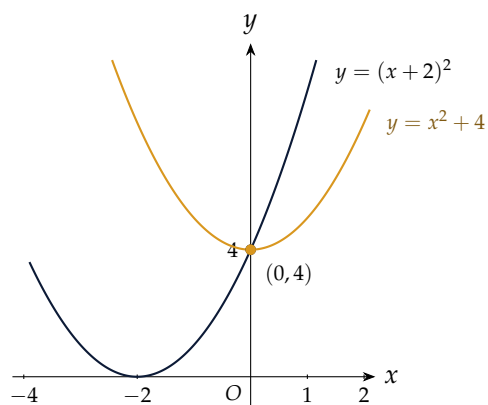
- Taking the square root of a sum term by term: from $r^2 + 4 = V$ you get $r = \sqrt{V-4}$, not $r = \sqrt{V} - 2$.
- Dividing only *part* of one side. If $ax = b + c$ then $x = \frac{b+c}{a}$, not $\frac{b}{a} + c$.
- Undoing operations in the wrong order. Always undo the *last* operation first.

7 Equations, identities and completing the square

An **equation** such as $(x+2)^2 = x^2 + 4$ is true for particular values of x only. Expand the left side: $x^2 + 4x + 4 = x^2 + 4$, so $4x = 0$ and the only solution is $x = 0$. The graphs of the two sides meet at exactly one point. An **identity** such as $(x+1)^2 \equiv x^2 + 2x + 1$ is true for *every* value of x : the two sides are the same expression written differently, so their graphs lie exactly on top of each other. The symbol $\equiv$ means "is identically equal to".



identity: the two curves coincide

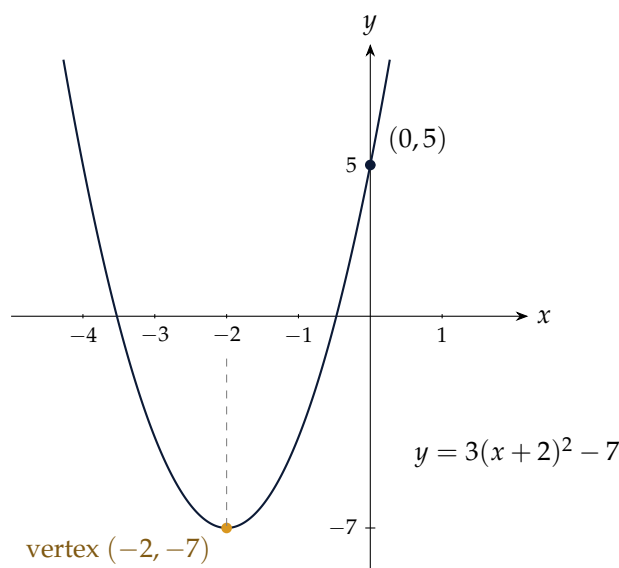


equation: true only at $x = 0$

To find unknown constants in an identity, expand one side and *compare coefficients*. If two polynomials are equal for all x , then the number multiplying each power of x must match. (Substituting two or three convenient values of x also works, since an identity must hold for every value.)

Completing the square. Any quadratic $x^2 + bx + c$ can be written as $(x+p)^2 + q$. This is an identity, and expanding shows $(x+p)^2 + q = x^2 + 2px + p^2 + q$. Comparing coefficients, $2p = b$ so $p = \frac{b}{2}$, and $q = c - p^2$. The quick rule: *halve*

the coefficient of x , then subtract its square. The form $a(x + p)^2 + q$ is useful because the graph of the quadratic has its turning point at $(-p, q)$: the square is never negative, so it is smallest, 0, when $x = -p$, and then the expression equals q .



Worked example: completing the square by comparing coefficients

Given that $3x^2 + 12x + 5 \equiv a(x + b)^2 + c$, find a , b and c .

Expand the right side: $a(x^2 + 2bx + b^2) + c = ax^2 + 2abx + (ab^2 + c)$.

Compare x^2 : $a = 3$. Compare x : $2ab = 12$, so $b = 2$. Compare constants: $ab^2 + c = 5$, so $12 + c = 5$ and $c = -7$.

So $3x^2 + 12x + 5 \equiv 3(x + 2)^2 - 7$. The graph above has its lowest point at $(-2, -7)$ and crosses the y -axis at $(0, 5)$; both are checked by substituting.

Worked example: an identity with an unknown

Given that $(x + 3)(x - 2) + k(x + 1) \equiv x^2 + 5x - 2$ for all x , find k .

Expand the left side: $x^2 + x - 6 + kx + k = x^2 + (1 + k)x + (k - 6)$.

Compare the coefficients of x : $1 + k = 5$, so $k = 4$. Check the constants: $k - 6 = -2$. ✓

Worked example: identity or equation?

Is $(x - 4)^2 - (x + 4)^2 = -16x$ an equation or an identity?

Expand: $x^2 - 8x + 16 - (x^2 + 8x + 16) = -16x$. The left side is $-16x$ for every x , so it is an *identity*. (Quicker: $A^2 - B^2 = (A - B)(A + B) = (-8)(2x) = -16x$.)

Common mistakes

- Forgetting that the constant term is $ab^2 + c$, not just c , when comparing coefficients for $a(x + b)^2 + c$.
- Believing that testing one value proves an identity. A single value can show an equation is true for that value, but only an algebraic argument shows an identity.
- Reading the vertex of $3(x + 2)^2 - 7$ as $(2, -7)$. The vertex is at $x = -2$.

8 Algebraic proof and reasoning

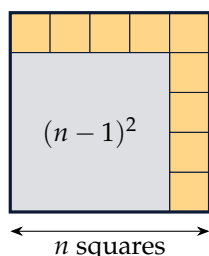
To *prove* that something is true for every number, an example is not enough: $3^2 - 2^2 = 5$ being odd does not prove anything about all squares. We write the numbers in general algebraic form, manipulate, and show the result has the required property.

Useful general forms. An even number is $2n$. An odd number is $2n + 1$. Consecutive integers are $n, n + 1, n + 2, \dots$. Consecutive even numbers are $2n, 2n + 2, \dots$. A multiple of 3 is $3n$. Use a *different* letter for each separate number, for example $2m + 1$ and $2n + 1$ for two odd numbers (using the same letter would force them to be equal).

The method. (1) Write the general forms. (2) Form the expression and expand it. (3) Factorise, and show a factor that gives the property: a factor 2 means even, a factor 3 means divisible by 3, a perfect square means it is a square.

Disproof. To show a claim is false, a single *counterexample* is enough.

The picture below proves a classical fact without algebra: a square of $n \times n$ tiles is made from an $(n-1) \times (n-1)$ square by adding an L-shaped border of n tiles along the top and $n-1$ down the side, so $n^2 - (n-1)^2 = 2n-1$. The difference between consecutive squares is always an odd number.



amber L-shape: n along the top
plus $n-1$ down the side
 $= 2n-1$ squares

$$\text{so } n^2 - (n-1)^2 = 2n-1$$

Worked example: product of two odd numbers

Prove that the product of two odd numbers is odd.

Let the numbers be $2m+1$ and $2n+1$ (different letters). Their product is $(2m+1)(2n+1) = 4mn + 2m + 2n + 1 = 2(2mn + m + n) + 1$. This is $2 \times (\text{an integer}) + 1$, which is odd. ■

Worked example: a perfect square from consecutive evens

Show that the product of two consecutive even numbers, plus 1, is always the square of an odd number.

The numbers are $2n$ and $2n+2$. Then $2n(2n+2) + 1 = 4n^2 + 4n + 1 = (2n+1)^2$, and $2n+1$ is odd. ■ (Check: $n=3$: $6 \times 8 + 1 = 49 = 7^2$.)

Worked example: a number trick

Pick any two-digit number, reverse its digits and subtract the smaller from the larger. Show the result is always a multiple of 9.

Let the digits be a (tens) and b (units) with $a > b$. The number is $10a+b$ and its reverse is $10b+a$. The difference is $(10a+b) - (10b+a) = 9a-9b = 9(a-b)$, a multiple of 9. ■

Worked example: symmetric expressions

Given that $x + \frac{1}{x} = 5$, find $x^2 + \frac{1}{x^2}$.

Square both sides: $\left(x + \frac{1}{x}\right)^2 = x^2 + 2 \times x \times \frac{1}{x} + \frac{1}{x^2} = x^2 + 2 + \frac{1}{x^2}$.

So $x^2 + 2 + \frac{1}{x^2} = 25$ and $x^2 + \frac{1}{x^2} = 23$. We never needed to find x .

Exam tip

Many ESAT questions give a relation such as $x + \frac{1}{x} = 5$ or $a+b=5$ and ask for a different combination. Do not try to find the numbers: square, cube or factorise the given relation until the wanted combination appears. The cube of $a+b$ is $a^3 + b^3 + 3ab(a+b)$, so $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$.

Common mistakes

- Using the same letter for two different numbers in a proof.
- Testing examples and calling it a proof. Examples can only disprove.
- Stopping before showing the property: finish by displaying the factor (for example $2 \times \dots$ for even).

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Find the value of $2x^2 - 3x$ when $x = -4$.
2. Expand and simplify $(2x - 5)^2$.
3. Factorise (a) $9x^2 - 49$ and (b) $2x^2 + x - 15$.
4. Simplify $\frac{x^2 - x - 12}{x^2 - 16}$.
5. Make a the subject of $s = \frac{a + b}{a - b}$.
6. Given $x^2 + 6x + 1 \equiv (x + p)^2 + q$, find p and q .
7. Without a calculator, find $\frac{5.1^2 - 4.9^2}{0.4}$.
8. Prove that the sum of three consecutive integers is a multiple of 3.

Answers to the self-check

1. $2(16) - 3(-4) = 32 + 12 = 44$.
2. $4x^2 - 20x + 25$.
3. (a) $(3x - 7)(3x + 7)$. (b) $ac = -30$, numbers 6 and -5 : $2x^2 + 6x - 5x - 15 = (2x - 5)(x + 3)$.
4. $\frac{(x-4)(x+3)}{(x-4)(x+4)} = \frac{x+3}{x+4}$.
5. $s(a-b) = a+b$, so $sa-sb = a+b$, so $a(s-1) = b(s+1)$ and $a = \frac{b(s+1)}{s-1}$.
6. $p = 3$ and $q = 1 - 9 = -8$.
7. $(5.1 - 4.9)(5.1 + 4.9) = 0.2 \times 10 = 2$, and $\frac{2}{0.4} = 5$.
8. $n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1)$.

Next step

Heurisco's *ESAT Maths 1: Algebra Practice Set* has twenty original exam-style questions on this lesson, with diagrams, in order of difficulty, with a full worked solution for each.