

ESAT

Maths 1

Angles, circles and Pythagoras

Free lesson notes

Covers

M5.1 to M5.10

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

Geometry is the part of the Maths 1 paper where a good picture and a few well-chosen facts replace pages of algebra. The facts themselves are not long, but they only work if you know *why* each one is true: then you can rebuild a half-remembered fact on the spot, and you can see which fact a question is hiding.

In this lesson every rule is derived from earlier ones, starting with the half-turn of a straight line. We build up through angles and parallel lines, polygons, congruence and similarity, Pythagoras' theorem (with a proof by rearranging squares), the vocabulary and basic properties of circles, and then the five circle theorems, each with a diagram and the reason it holds. The last two sections cover distance and midpoints on a coordinate grid (M5.10) and applied reasoning with clocks, ladders and shadows.

How to use it. Cover each worked example and try it yourself first. In every angle question write the *reason* beside each step ("angles on a straight line", "alternate angles", and so on): it is the quickest way to catch your own errors. Diagrams in exam questions are usually *not drawn to scale*, so never measure; calculate. Specification codes covered: M5.1 to M5.10.

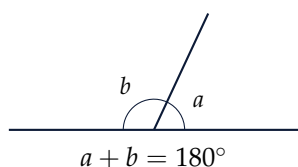
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1 Angles and parallel lines

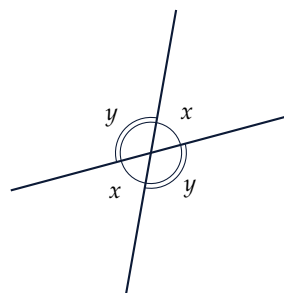
Words you need. An *angle* measures an amount of turn at a point (the *vertex*) between two lines. We use degrees: one full turn is 360° , so a half-turn (a straight line) is 180° and a quarter-turn (a right angle) is 90° . An angle is *acute* if it is less than 90° , *obtuse* if it is between 90° and 180° , and *reflex* if it is more than 180° . Two lines are *perpendicular* if they meet at a right angle and *parallel* if they never meet (we mark them with matching arrowheads). To *bisect* is to cut exactly in half. We name an angle with three letters, the middle one being the vertex: $\angle ABC$ is the angle at B.

Three basic facts.

- **Angles on a straight line add to 180°** (together they make a half-turn).
- **Angles around a point add to 360°** (a full turn).
- **Vertically opposite angles are equal.** Reason: in the diagram, $x + y = 180^\circ$ (straight line) and, on the other line, $y + x' = 180^\circ$. Both x and x' equal $180^\circ - y$, so $x = x'$.



angles on a straight line

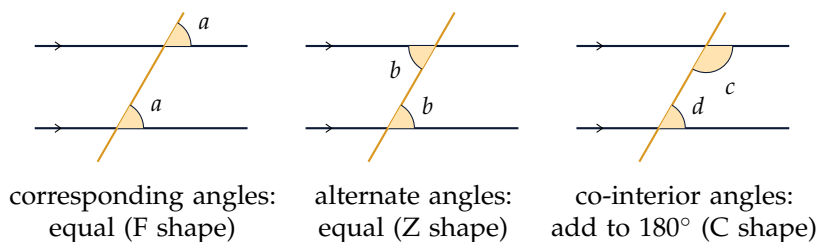


vertically opposite angles are equal; $x + y = 180^\circ$

Parallel lines. A line crossing a pair of parallel lines (a *transversal*) makes eight angles, and only two different sizes occur. The key idea is that the lower parallel line is just the upper one *slid along the transversal*, so each angle at the top has an identical twin at the bottom. This gives the first fact, and the others follow from it with the basic facts above.

- **Corresponding angles are equal** ("F shape"): the sliding argument.
- **Alternate angles are equal** ("Z shape"): a corresponding angle equals the angle vertically opposite it, so the two alternate angles are equal.

- **Co-interior angles add to 180°** ("C shape"): one of them is 180° minus an angle equal to the other (corresponding, then straight line).



These facts work in both directions: if a pair of corresponding or alternate angles are equal, the lines *must* be parallel.

Worked example: angles around a point

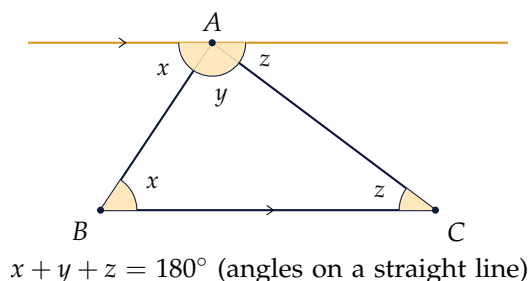
Four angles meet at a point: 90° , $(2x + 10)^\circ$, $3x^\circ$ and $(x + 50)^\circ$. Find x .

They fill a full turn: $90 + 2x + 10 + 3x + x + 50 = 360$, so $6x + 150 = 360$ and $x = 35$. The angles are 90° , 80° , 105° , 85° , which do add to 360° .

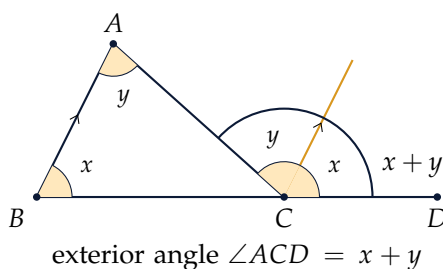
Worked example: parallel lines

A transversal crosses two parallel lines. One of the angles it makes with the upper line is 65° . Find the other seven. The vertically opposite angle is 65° . Alternate and corresponding angles at the lower line give 65° twice more there, so four angles equal 65° . Every other angle is next to one of these on a straight line, so it is $180 - 65 = 115^\circ$ (co-interior angles add to 180°). There are four angles of each size.

The angles of a triangle add to 180° . Draw the line through the top vertex A parallel to the base BC . The two angles it makes with the sides at A are alternate angles to the base angles at B and C (the same letter x and z in the diagram). Together with the angle y at A they lie on a straight line, so $x + y + z = 180^\circ$.



Exterior angle of a triangle. Extend one side. The angle formed outside the triangle is the *exterior angle*. It equals the sum of the two interior angles opposite it. Reason: the exterior angle and the adjacent interior angle make a straight line, so exterior = $180^\circ - (\text{adjacent interior}) = \text{the sum of the other two}$ (as all three interior angles add to 180°). In the diagram the line from C is drawn parallel to BA , which shows the same thing with corresponding and alternate angles.



Worked example: triangle facts

In triangle PQR , $\angle P = 50^\circ$ and $\angle Q = 60^\circ$, and QR is extended to S . Find $\angle PRS$.

$\angle R = 180 - 50 - 60 = 70^\circ$, so $\angle PRS = 180 - 70 = 110^\circ$. Or directly: the exterior angle is $50 + 60 = 110^\circ$.

Common mistakes

- Using the wrong parallel-line fact: look for the shape. Equal angles come from F and Z; a C shape adds to 180° .
- Assuming two lines are parallel because they *look* parallel. Only the arrowheads or a stated fact allow you to use the rules.
- Measuring from the diagram. It is not to scale.

2 Polygons, triangles and quadrilaterals

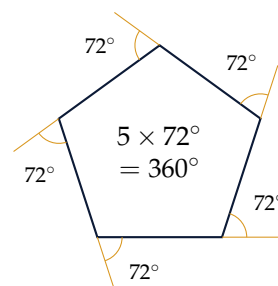
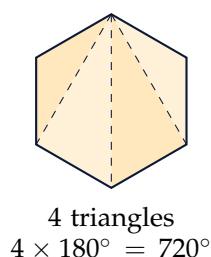
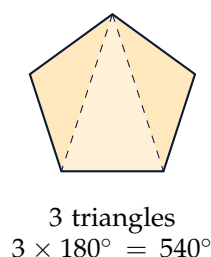
A *polygon* is a closed shape with straight sides; it is *regular* if all its sides and all its angles are equal. At each corner, the *interior angle* is inside the shape and the *exterior angle* is the extra turn needed when you extend a side, so interior + exterior = 180° .

Triangles. An *equilateral* triangle has three equal sides and so three angles of 60° (shown below). An *isosceles* triangle has two equal sides, and the angles opposite them (the base angles) are equal; we prove this in Section 3. A *scalene* triangle has no equal sides; a *right-angled* triangle has one 90° angle.

Interior angles of a polygon. Pick one vertex and join it to all the others. This cuts an n -sided polygon into $n - 2$ triangles, and the angles of those triangles exactly make up the angles of the polygon. So

$$\text{sum of interior angles} = (n - 2) \times 180^\circ.$$

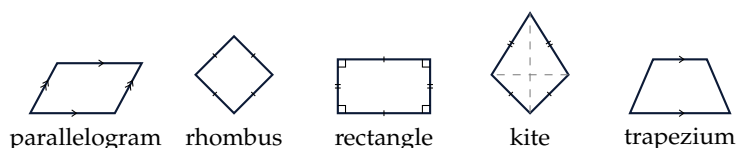
Exterior angles. Imagine walking round the polygon: at each corner you turn through the exterior angle, and when you get back to the start you have turned exactly once round, 360° . So *the exterior angles of any polygon add to 360°* . For a regular polygon the n exterior angles are equal, so each is $360^\circ \div n$, and the interior angle is 180° minus that.



Worked example: regular polygons

- (a) Find the interior angle of a regular octagon. (b) A regular polygon has interior angles of 156° . How many sides?
 (a) Exterior angle = $360 \div 8 = 45^\circ$, so the interior angle is $180 - 45 = 135^\circ$. (Check: $(8 - 2) \times 180 = 1080 = 8 \times 135$.)
 (b) The exterior angle is $180 - 156 = 24^\circ$, so $n = 360 \div 24 = 15$ sides.

Quadrilaterals. A diagonal splits a quadrilateral into two triangles, so its interior angles add to $2 \times 180 = 360^\circ$. The special quadrilaterals are defined by their sides, and their other properties follow:



- **Parallelogram:** opposite sides parallel and equal; opposite angles equal; adjacent angles add to 180° (co-interior angles); diagonals bisect each other.
- **Rhombus:** a parallelogram with four equal sides; its diagonals cross at right angles and bisect the angles at the corners.
- **Rectangle:** four right angles; the diagonals are equal and bisect each other. A **square** is both a rectangle and a rhombus.
- **Kite:** two pairs of equal adjacent sides; one diagonal is a line of symmetry, and the diagonals cross at right angles.
- **Trapezium:** one pair of parallel sides.

Worked example: a kite

$ABCD$ is a kite with $AB = AD$ and $CB = CD$. $\angle BAD = 70^\circ$ and $\angle ABC = 110^\circ$. Find $\angle BCD$.

The kite is symmetrical about AC , so $\angle ADC = \angle ABC = 110^\circ$. The four angles add to 360° , so $\angle BCD = 360 - 70 - 110 - 110 = 70^\circ$.

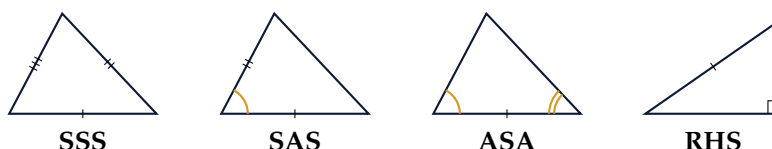
Common mistakes

- Using $(n - 2) \times 180$ for the *exterior* angles (they always total 360°) or $360 \div n$ for the interior angle.
- For a regular polygon, giving the exterior angle when the question asks for the interior (or the other way round).
- Believing a rhombus has equal diagonals: that is a rectangle. The rhombus diagonals are perpendicular.

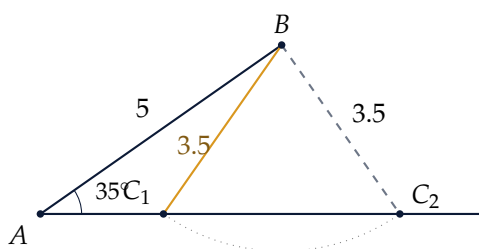
3 Congruence, similarity and transformations

Two shapes are **congruent** if one can be moved onto the other exactly (same shape *and* size, possibly turned over). Two triangles are congruent if you can show *any one* of these four conditions:

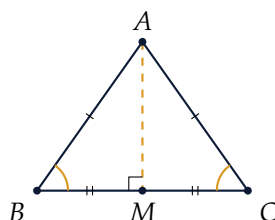
- SSS**: three pairs of equal sides.
- SAS**: two pairs of equal sides and the equal angle *between* them.
- ASA** (or **AAS**): two pairs of equal angles and a pair of equal sides.
- RHS**: right angle, hypotenuse and one other side equal.



Why “SSA” is not enough. Given two sides and an angle that is *not* between them, there can be two different triangles. Draw the angle at A , mark B on one ray, and swing an arc of radius BC round B : it may cut the other ray in two places, giving two triangles ABC_1 and ABC_2 that are not congruent. (RHS is the safe special case where the angle is 90° : the arc then touches the ray in only one place.)

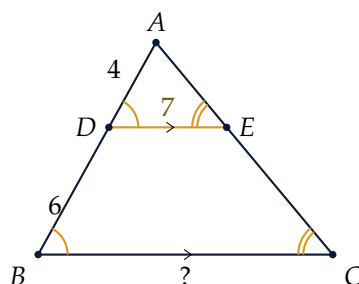


Using congruence: the base angles of an isosceles triangle are equal. Let $AB = AC$ and let M be the midpoint of BC . Triangles ABM and ACM have $AB = AC$, $BM = CM$ and the side AM in common, so they are congruent (SSS). Matching up the corresponding angles gives $\angle ABM = \angle ACM$. The same congruence shows $\angle AMB = \angle AMC$, and as these make a straight line each is 90° : the line from the apex to the midpoint of the base is perpendicular to it.



Similar shapes. Shapes are **similar** if one is an enlargement of the other: all the angles are equal and all the lengths are in the same ratio, the *scale factor*. This gives the most useful technique in the whole topic: *match up the corresponding sides, write down a ratio, and solve.*

A line drawn parallel to the base of a triangle always cuts off a similar triangle: the angles at D and E equal those at B and C (corresponding angles), and the angle at A is shared.



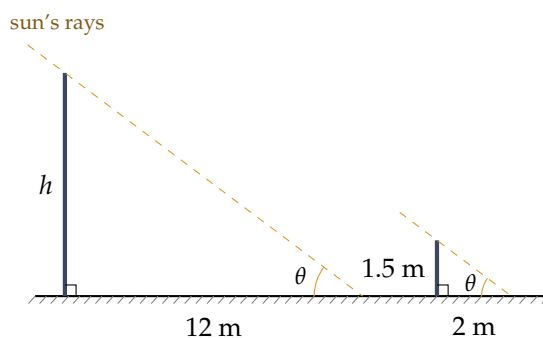
Worked example: similar triangles

In the triangle above $DE \parallel BC$, $AD = 4$, $DB = 6$ and $DE = 7$. Find BC .

Triangle ADE is similar to ABC . The scale factor from small to large is $\frac{AB}{AD} = \frac{10}{4} = 2.5$, so $BC = 7 \times 2.5 = 17.5$.

Worked example: shadows

At the same time of day a 1.5 m stick casts a 2 m shadow and a tree casts a 12 m shadow. How tall is the tree?

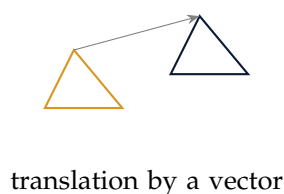
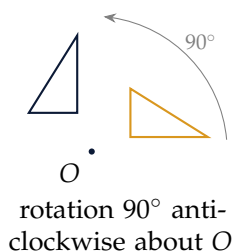
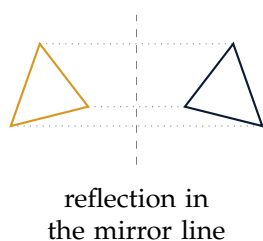


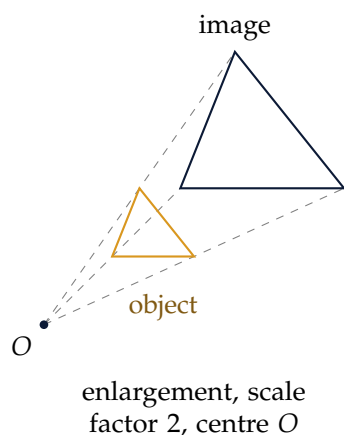
The sun's rays are parallel, so the two right-angled triangles have the same angles and are similar. The tree is $12 \div 2 = 6$ times as far as the stick's shadow, so it is $6 \times 1.5 = 9$ m tall.

Transformations. A transformation moves a shape. To describe one completely you must give all the details:

- **Reflection:** the mirror line (each point and its image are the same distance from it, on opposite sides).
- **Rotation:** the centre, the angle and the direction (clockwise or anticlockwise).
- **Translation:** a vector $\begin{pmatrix} a \\ b \end{pmatrix}$ meaning a to the right and b up.
- **Enlargement:** the centre and the scale factor. Each image point lies on the line from the centre through the object point, at k times the distance.

Reflections, rotations and translations produce congruent shapes; enlargements produce similar shapes.





Worked example: an enlargement

Enlarge the point $P(2,3)$ by scale factor 3 about the centre $C(1,1)$.
 P is 1 right and 2 up from C . The image is 3 times as far: 3 right and 6 up from C , which is $(4,7)$.

Common mistakes

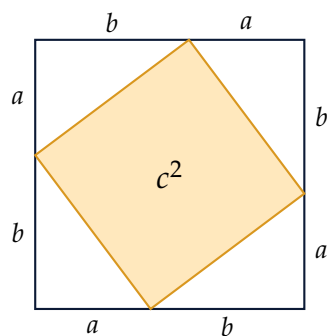
- Using SSA as a congruence test, or forgetting that the angle in SAS must be *between* the two sides.
- Matching the wrong sides when writing a ratio. Name the triangles with the vertices in corresponding order (ADE and ABC) and read off the pairs.
- A rotation described without a centre, or an enlargement without a centre, is incomplete.

4 Pythagoras' theorem and distance

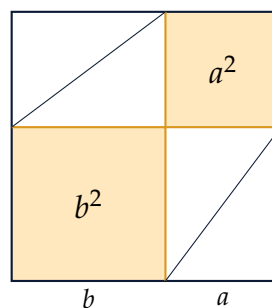
In a right-angled triangle the side opposite the right angle is the longest and is called the **hypotenuse**. **Pythagoras' theorem** says that if the other two sides are a and b and the hypotenuse is c ,

$$a^2 + b^2 = c^2.$$

Why it is true. Take four copies of the right-angled triangle and arrange them inside a square of side $a + b$. In the left picture the four triangles leave a tilted square of side c in the middle, so $(a + b)^2 = c^2 + 4\left(\frac{1}{2}ab\right) = c^2 + 2ab$. In the right picture the same four triangles are arranged in pairs to make two $a \times b$ rectangles, which leave two squares of areas a^2 and b^2 in the *same big square*, so $(a + b)^2 = a^2 + b^2 + 2ab$. Both equal the area of the big square, so $c^2 + 2ab = a^2 + b^2 + 2ab$, and $c^2 = a^2 + b^2$.



four triangles and
a tilted square

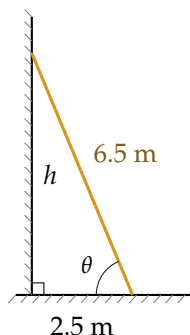


same triangles, two squares

Using it. To find a hypotenuse, square, add, square-root. To find a shorter side, *subtract*: $a^2 = c^2 - b^2$. Whole-number solutions (Pythagorean triples) are worth knowing on sight: 3, 4, 5; 5, 12, 13; 8, 15, 17; 7, 24, 25; and their multiples such as 6, 8, 10. The theorem also works backwards: if three sides satisfy $a^2 + b^2 = c^2$, the triangle has a right angle.

Worked example: a ladder

A 6.5 m ladder stands with its foot 2.5 m from a vertical wall. How high up the wall does it reach?



The ladder, wall and ground form a right-angled triangle with hypotenuse 6.5. So $h^2 = 6.5^2 - 2.5^2 = 42.25 - 6.25 = 36$ and $h = 6$ m. (It is a 5, 12, 13 triangle scaled by $\frac{1}{2}$.)

Worked example: is it right-angled?

Is a triangle with sides 8, 15 and 17 right-angled?

The longest side is 17. $8^2 + 15^2 = 64 + 225 = 289 = 17^2$, so yes, with the right angle opposite the side 17.

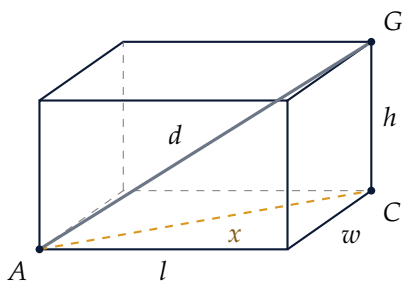
Worked example: an isosceles triangle

An isosceles triangle has sides 13, 13 and 10. Find its area.

Draw the perpendicular from the apex to the base. It bisects the base (Section 3), giving a right-angled triangle with hypotenuse 13 and base 5. The height is $\sqrt{13^2 - 5^2} = 12$, so the area is $\frac{1}{2} \times 10 \times 12 = 60$.

Three dimensions. In a cuboid with edges l , w and h , the diagonal across the base has length x with $x^2 = l^2 + w^2$. The space diagonal d from a bottom corner to the opposite top corner forms a second right-angled triangle with x and h , so $d^2 = x^2 + h^2$, giving

$$d = \sqrt{l^2 + w^2 + h^2}.$$

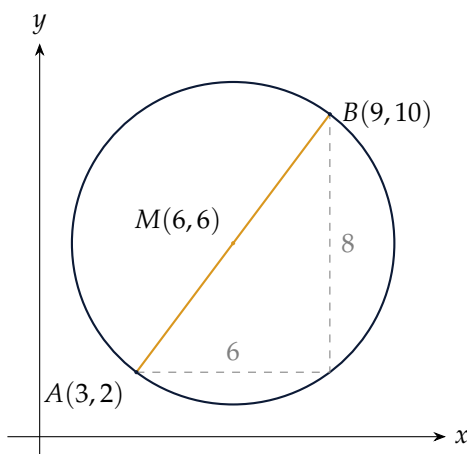


Worked example: the space diagonal

A box measures 3 by 4 by 12. What is the longest straight rod that fits inside it?

$$d = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{169} = 13.$$

Distance on a coordinate grid (M5.10). The distance between (x_1, y_1) and (x_2, y_2) is the hypotenuse of a right-angled triangle with sides $|x_2 - x_1|$ and $|y_2 - y_1|$, so it equals $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. The midpoint is the average of each coordinate, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, and the gradient is $\frac{y_2 - y_1}{x_2 - x_1}$.



Worked example: distance and midpoint

$A(3,2)$ and $B(9,10)$. Find AB and the midpoint of AB .

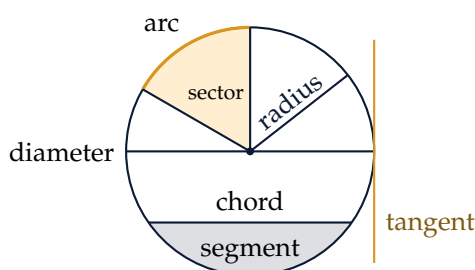
The differences are 6 and 8, so $AB = \sqrt{36 + 64} = 10$. The midpoint is $\left(\frac{3+9}{2}, \frac{2+10}{2}\right) = (6,6)$. These are the diameter and centre of the circle drawn.

Common mistakes

- Adding where you should subtract: a shorter side is $\sqrt{c^2 - b^2}$.
- Applying $a^2 + b^2 = c^2$ to a triangle with no right angle.
- Forgetting the final square root, or squaring a negative difference incorrectly (the minus sign disappears when squared).

5 Circles: terms and first properties

The **circle** is the set of all points a fixed distance (the **radius**) from a fixed point (the **centre**). The vocabulary matters because questions are written using it: the **circumference** is the distance round; a **diameter** is a chord through the centre (twice the radius); a **chord** joins two points of the circle; an **arc** is a part of the circumference; a **sector** is the region between two radii and an arc; a **segment** is the region between a chord and an arc; a **tangent** is a line that touches the circle at exactly one point.



Because every radius has the same length, any triangle with two radii as sides is isosceles. This one observation drives almost all circle-theorem proofs.

The line from the centre perpendicular to a chord bisects it. Let O be the centre, AB a chord, and M the foot of the perpendicular from O . In the right-angled triangles OMA and OMB the hypotenuses are both radii, and OM is common, so they are congruent (RHS) and $AM = MB$. The converse is also true: a line from the centre to the *midpoint* of a chord is perpendicular to it. Pythagoras then gives $r^2 = d^2 + \left(\frac{1}{2}\text{chord}\right)^2$, where $d = OM$.

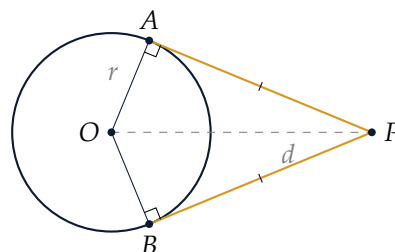
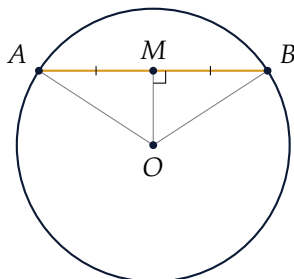
Worked example: distance to a chord

A chord of length 16 lies in a circle of radius 10. How far is the chord from the centre?

The half-chord is 8, so $d^2 = 10^2 - 8^2 = 36$ and $d = 6$.

A tangent is perpendicular to the radius at the point of contact. Reason: the tangent touches only at A , so every other point on it is *outside* the circle, farther from O than A is. So A is the closest point of the line to O , and the shortest distance from a point to a line is the perpendicular one.

Tangents from an external point are equal in length. Draw the two tangents from P touching at A and B . The right-angled triangles OAP and OBP have the same hypotenuse OP and equal sides $OA = OB$ (radii), so they are congruent (RHS) and $PA = PB$. By Pythagoras, $PA = \sqrt{OP^2 - r^2}$. The figure $OAPB$ is a kite.



Worked example: tangent length

P is 13 cm from the centre of a circle of radius 5 cm. Find the length of a tangent from P , and the area of the kite $OAPB$.

$PA = \sqrt{13^2 - 5^2} = 12$. The kite is two right-angled triangles, each of area $\frac{1}{2} \times 5 \times 12 = 30$, so its area is 60 cm^2 .

Common mistakes

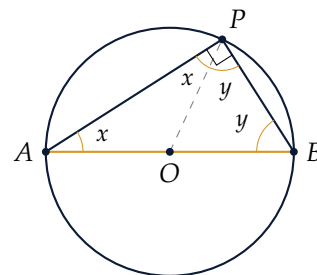
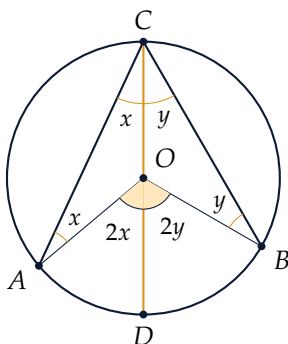
- Treating a tangent as if it passed through the centre: it is *perpendicular* to the radius.
- Using $r + d$ or $d - r$ instead of $\sqrt{d^2 - r^2}$ for the tangent length.
- Forgetting to double the half-chord when the question asks for the whole chord.

6 The circle theorems, with reasons

Each theorem below is proved from what you already know: isosceles triangles (equal radii), the exterior-angle fact and the angle sum. Say the reason in full when you use one: "angle at the centre is twice the angle at the circumference".

Theorem 1: the angle at the centre is twice the angle at the circumference (when both stand on the same arc). Draw CO and extend it to D . Triangle OAC is isosceles ($OA = OC$), so its base angles are equal; call them x . The exterior angle $\angle AOD$ of triangle OAC equals the sum of the two interior angles opposite it: $x + x = 2x$. In the same way $\angle BOD = 2y$ where $y = \angle OCB$. So $\angle AOB = 2x + 2y = 2(x + y) = 2\angle ACB$.

Theorem 2: the angle in a semicircle is a right angle. If AB is a diameter, the angle at the centre on AB is 180° , so by Theorem 1 the angle at the circumference is 90° . Directly: $OA = OP = OB$ makes two isosceles triangles with base angles x and y , and the angles of triangle APB are $x + y + (x + y) = 180^\circ$, so $x + y = 90^\circ$. The converse is useful: the hypotenuse of a right-angled triangle is a diameter of its circumscribed circle.

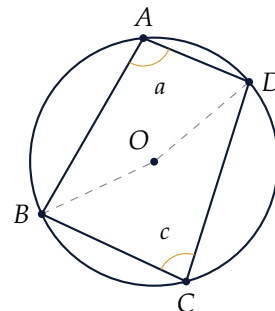
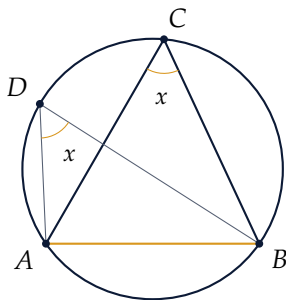


Worked example: angle in a semicircle

AB is a diameter and C is on the circle with $\angle CAB = 27^\circ$. Find $\angle ABC$.
 $\angle ACB = 90^\circ$, so $\angle ABC = 180 - 90 - 27 = 63^\circ$.

Theorem 3: angles in the same segment are equal. Two points C and D on the same side of the chord AB both see AB at the same angle, because each is half of the angle at the centre on AB (Theorem 1).

Theorem 4: opposite angles of a cyclic quadrilateral add to 180° . (A *cyclic* quadrilateral has all four vertices on a circle.) Take the diagonal BD . The angle a at A stands on the arc BCD , and the angle at the centre on the same arc is $2a$. The angle c at C stands on the other arc BAD , whose centre angle (the reflex angle) is $2c$. The two centre angles make a full turn: $2a + 2c = 360^\circ$, so $a + c = 180^\circ$. A consequence: an *exterior* angle of a cyclic quadrilateral equals the interior angle opposite it.



Worked example: a cyclic quadrilateral

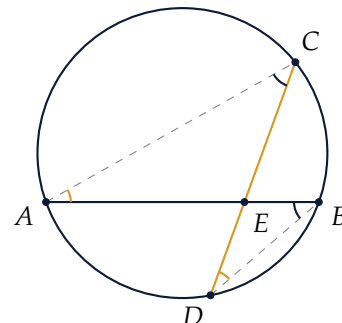
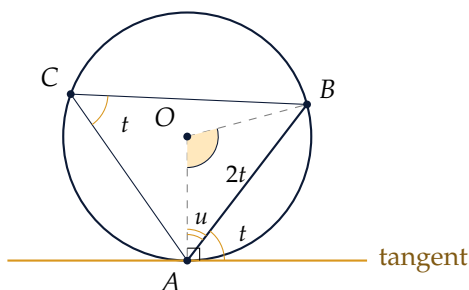
$ABCD$ is cyclic with $\angle ABC = 104^\circ$ and $\angle CAD = 30^\circ$. Find $\angle ADC$ and $\angle ACD$.
 Opposite angles add to 180° , so $\angle ADC = 180 - 104 = 76^\circ$. Then in triangle ACD , $\angle ACD = 180 - 76 - 30 = 74^\circ$.

Theorem 5: the alternate segment theorem. The angle between a tangent and a chord equals the angle in the alternate (opposite) segment. Reason: let the angle between the tangent at A and the chord AB be t . The radius OA is perpendicular to the tangent, so $\angle OAB = 90^\circ - t$. Triangle OAB is isosceles, so $\angle OBA = 90^\circ - t$ too, and the angle at the centre is $\angle AOB = 180 - 2(90 - t) = 2t$. By Theorem 1 the angle at the circumference on the alternate side is half of that, t .

Worked example: tangent and chord

The tangent at A and the chord AB make an angle of 54° . Find the angle at the centre $\angle AOB$ and the base angles of triangle OAB .
 The angle at the centre is $2 \times 54 = 108^\circ$, and the base angles are $(180 - 108) \div 2 = 36^\circ$ (which is $90 - 54$, as expected).

A useful by-product: crossing chords. If chords AB and CD cross at E , triangles AEC and DEB have equal angles (angles in the same segment on AC and on BD , plus vertically opposite angles at E), so they are similar. Matching the sides gives $\frac{AE}{DE} = \frac{CE}{BE}$, that is $AE \times EB = CE \times ED$.



Worked example: crossing chords

Chords AB and CD cross at E with $AE = 8$, $EB = 3$ and $CD = 10$, where $CE > ED$. Find CE .
 $AE \times EB = 24$, so $CE \times ED = 24$ with $CE + ED = 10$. Let $CE = u$: $u(10 - u) = 24$, $u^2 - 10u + 24 = 0$, $(u - 4)(u - 6) = 0$. Since $CE > ED$, $CE = 6$.

Exam tip

In a circle problem, *mark every radius*. Equal radii give isosceles triangles, and an isosceles triangle gives two equal angles and often a hidden $2\times$ relationship. Then ask: is there a diameter (right angle)? a tangent (right angle with the radius)? four points on a circle (cyclic quadrilateral)?

Common mistakes

- Using the angle-at-centre theorem when the two angles do not stand on the *same arc*.
- Applying the cyclic-quadrilateral rule to *adjacent* angles: it is the *opposite* angles that add to 180° .
- In the alternate segment theorem, picking the angle in the *same* side as the angle between tangent and chord. The alternate segment is on the other side of the chord.

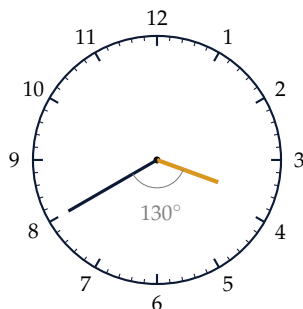
7 Applied geometry: clocks, ladders and shadows

ESAT likes questions that use geometry in everyday settings. The method is always the same: draw the situation, spot the right-angled triangles or the similar triangles, and use the facts above.

Clocks. The minute hand turns through 360° in 60 minutes, which is 6° per minute. The hour hand turns 360° in 12 hours, which is 30° per hour or 0.5° per minute. At h hours and m minutes, measured clockwise from 12,

$$\text{hour hand} = 30h + 0.5m, \quad \text{minute hand} = 6m.$$

The angle between the hands is the difference of these two numbers (if it comes out above 180° , subtract it from 360°).



Worked example: angle between the hands

Find the angle between the hands at 3:40 (shown above).

Hour hand: $30 \times 3 + 0.5 \times 40 = 110^\circ$. Minute hand: $6 \times 40 = 240^\circ$. The angle between them is $240 - 110 = 130^\circ$.

Worked example: when do the hands meet?

The hands of a clock coincide at 12:00. When do they next coincide?

The minute hand gains on the hour hand at $6 - 0.5 = 5.5^\circ$ per minute. It must gain a full turn, so it takes $360 \div 5.5 = \frac{720}{11} = 65\frac{5}{11}$ minutes. They coincide every $65\frac{5}{11}$ minutes, next at about 1:05:27. (This happens 11 times in 12 hours, not 12.)

Ladders and walls. A ladder against a wall is the hypotenuse of a right-angled triangle, so Pythagoras applies. The surprising feature is that the foot and the top do not move equally: if the top slips down by 0.4 m, the foot moves out by *more* than 0.4 m, which you cannot tell without calculating.

Shadows and heights. The sun's rays are parallel, so a vertical object and its shadow form a right-angled triangle similar to that of any other vertical object at the same time. Ratio $\frac{\text{height}}{\text{shadow}}$ is the same for both.

Spotting the structure. If a question gives you a ratio of lengths, think similar triangles. If it gives you lengths and a right angle, think Pythagoras. If it gives angles and a circle, think theorems and equal radii. If it asks about time, convert to degrees.

Exam tip

Before calculating anything, ask whether the answer must be bigger or smaller than a number you can see. For example, a ladder's foot moves further out than the top moves down, and the angle between clock hands can never exceed 180° when you take the smaller angle. These sanity checks catch most slips.

8 Strategy: how to chase angles

A routine. (1) Mark any angle you are told and anything marked equal or parallel. (2) Find an angle you can write straight away: straight line, point, triangle, opposite, parallel. (3) Write it on the diagram *with the reason*, and continue. (4) If you are stuck, add a line: join the centre to a vertex, drop a perpendicular, extend a side, or draw a parallel. (5) Check that all the angles at a point or in a triangle add to the right total.

When algebra is given. If angles are written as expressions in x , find the equation first from the geometry (a straight line, a triangle, a cyclic quadrilateral) and solve it; then substitute back to find the required angle, which is often *not* x .

When a question names a shape ("a regular hexagon", "a rhombus"), write down all of its properties, since the hidden facts are usually the equal sides and symmetry. In a regular hexagon each interior angle is 120° , the main diagonals pass through the centre, and the triangle formed by two neighbouring vertices and the centre is equilateral.

Worked example: a hexagon

$ABCDEF$ is a regular hexagon. Find $\angle ACD$.

Each interior angle is 120° . Triangle ABC is isosceles with apex angle 120° , so $\angle BCA = 30^\circ$ and $\angle ACD = 120 - 30 = 90^\circ$. (Alternatively AD passes through the centre, so it is a diameter, and the angle in a semicircle is 90° .)

Check yourself

Try these without looking back, then compare with the answers on the next page.

- Find the interior angle of a regular polygon with 20 sides.
- A transversal crosses two parallel lines making an angle of 72° with the upper line. What is the co-interior angle on the lower line?
- A 10 m ladder reaches 8 m up a wall. How far is its foot from the wall?
- Triangles ABC and ADE are similar with $DE \parallel BC$, $AD = 3$, $AB = 9$ and $DE = 5$. Find BC .
- The angle at the centre of a circle on an arc is 124° . What is the angle at the circumference on the same arc?
- A cyclic quadrilateral has angles 85° and 112° at adjacent vertices. Find the other two.
- A tangent of length 24 is drawn to a circle from a point 25 from the centre. Find the radius.
- Find the angle between the clock hands at 5:20.

Answers to the self-check

1. Exterior angle $360 \div 20 = 18^\circ$, so the interior angle is 162° .
2. Co-interior angles add to 180° , so 108° .
3. $\sqrt{100 - 64} = 6$ m.
4. Scale factor $9 \div 3 = 3$, so $BC = 15$.
5. Half of it: 62° .
6. The opposite angles add to 180° : 95° opposite the 85° angle, and 68° opposite the 112° angle.
7. $r = \sqrt{25^2 - 24^2} = 7$.
8. Hour: $150 + 10 = 160^\circ$. Minute: 120° . The angle is 40° .

Next step

Heurisco's *ESAT Maths 1: Angles, Circles and Pythagoras Practice Set* has twenty original exam-style questions on this lesson, with diagrams, in order of difficulty, with a full worked solution for each.