

ESAT

Maths 1

Area, volume, trigonometry and vectors

Free lesson notes

Covers

M5.11 to M5.19

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

This lesson covers the measurement and geometry of shapes: area and perimeter (including arcs and sectors), volume and surface area of solids, right-angled trigonometry with exact values, bearings and vectors. Questions are usually exact: answers are given in terms of π or with surds, so you should leave π in your working and avoid decimals.

How to use it. Try each worked example first. Then do the self-check, then the practice set. Formulae for the sphere, cone and pyramid are provided on the ESAT, but you still need to know which to use and how to find the missing length (usually with Pythagoras), so learn them anyway.

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1 Area, arcs and sectors

Areas to know

- Triangle $\frac{1}{2}bh$; parallelogram bh ; trapezium $\frac{1}{2}(a+b)h$.
- Circle: area πr^2 , circumference $2\pi r$.
- Sector of angle θ : arc = $\frac{\theta}{360} \times 2\pi r$ and area = $\frac{\theta}{360} \times \pi r^2$.

The perimeter of a sector includes the two radii as well as the arc.

Worked example: a sector

A sector has radius 12 cm and angle 150° . Find its arc length and area.

Fraction of the circle: $\frac{150}{360} = \frac{5}{12}$.

Arc = $\frac{5}{12} \times 2\pi \times 12 = 10\pi$ cm. Area = $\frac{5}{12} \times \pi \times 144 = 60\pi$ cm².

Common mistakes

- Leaving the two radii out of the perimeter of a sector.
- Using the diameter in πr^2 .

2 Volume and surface area

- Prism: volume = (area of cross-section) $\times$ length.
- Cylinder: $V = \pi r^2 h$; curved surface area $2\pi r h$.
- Cone: $V = \frac{1}{3}\pi r^2 h$ with h the *vertical* height; curved surface $\pi r l$ where l is the slant height, $l^2 = r^2 + h^2$.
- Sphere: $V = \frac{4}{3}\pi r^3$; surface area $4\pi r^2$.
- Pyramid: $V = \frac{1}{3} \times \text{base area} \times \text{vertical height}$.

Worked example: a cone

A cone has base radius 6 cm and slant height 10 cm. Find its volume.

Vertical height: $h = \sqrt{10^2 - 6^2} = 8$. $V = \frac{1}{3}\pi(36)(8) = 96\pi$ cm³.

Worked example: a composite solid

A solid is a hemisphere of radius 3 on top of a cone with the same base radius and height 4. Find its volume.

Hemisphere: $\frac{2}{3}\pi(27) = 18\pi$. Cone: $\frac{1}{3}\pi(9)(4) = 12\pi$. Total = 30π .

Common mistakes

- Using the slant height in the cone volume formula. The volume needs the vertical height.
- Using a full sphere's formula for a hemisphere.

3 Right-angled trigonometry and exact values

In a right-angled triangle: $\sin \theta = \frac{\text{opp}}{\text{hyp}}$, $\cos \theta = \frac{\text{adj}}{\text{hyp}}$, $\tan \theta = \frac{\text{opp}}{\text{adj}}$ (SOH-CAH-TOA). You must know the exact values:

θ	0°	30°	45°	60°	90°
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	—

Draw an equilateral triangle of side 2 (halved) or a square of side 1 to rebuild them quickly.

Worked example: exact values

Find the height of an equilateral triangle of side 10.

Halve it: a right triangle with hypotenuse 10 and angle 60° . Height = $10 \sin 60^\circ = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}$.

Worked example: a missing ratio

A triangle has $\tan \theta = \frac{5}{12}$. Find $\sin \theta$.

Opposite 5, adjacent 12, so the hypotenuse is $\sqrt{25 + 144} = 13$ and $\sin \theta = \frac{5}{13}$.

Common mistakes

- Choosing the wrong pair of sides: label opposite, adjacent and hypotenuse relative to the angle first.
- Mixing up $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

4 Bearings

A bearing is measured *clockwise from north*, with three figures (for example 045° or 270°). To find a back-bearing, add or subtract 180° . Draw a north line at each point and mark angles carefully.

Worked example: two legs of a journey

A ship sails 10 km on a bearing of 060° from A to B, then 10 km on a bearing of 150° to C. Find the bearing of C from A.

The bearing of A from B is $060 + 180 = 240^\circ$. The angle $ABC = 240 - 150 = 90^\circ$, so triangle ABC is an isosceles right-angled triangle and $\angle BAC = 45^\circ$.

C is a further 45° clockwise from B as seen from A: $060 + 45 = 105^\circ$.

Common mistakes

- Measuring the bearing anticlockwise, or from the wrong north line.
- Writing a bearing with fewer than three figures (it is 060° , not 60°).

5 Vectors

A vector can be written in column form $\begin{pmatrix} x \\ y \end{pmatrix}$ or as $\vec{AB}$. You add and subtract components, and multiply by a scalar. The magnitude is $\sqrt{x^2 + y^2}$. Reversing direction negates the vector: $\vec{BA} = -\vec{AB}$. In geometry problems, write each unknown vector as a *route* along known vectors.

Worked example: column vectors

$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$. Find $|2\mathbf{a} - 3\mathbf{b}|^2$.

$2\mathbf{a} - 3\mathbf{b} = \begin{pmatrix} 4 + 3 \\ 6 - 12 \end{pmatrix} = \begin{pmatrix} 7 \\ -6 \end{pmatrix}$, so $|2\mathbf{a} - 3\mathbf{b}|^2 = 49 + 36 = 85$.

Worked example: vector geometry

$\vec{OA} = \mathbf{a}$ and $\vec{OB} = \mathbf{b}$. M is the midpoint of AB and P lies on OM with $OP : PM = 2 : 1$. Find $\vec{AP}$.

$\vec{OM} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\vec{OP} = \frac{2}{3}\vec{OM} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$.

$\vec{AP} = \vec{AO} + \vec{OP} = -\mathbf{a} + \frac{1}{3}(\mathbf{a} + \mathbf{b}) = \frac{1}{3}(\mathbf{b} - 2\mathbf{a})$.

Common mistakes

- Getting a vector backwards: $\vec{AB} = \mathbf{b} - \mathbf{a}$ means “end minus start”.
- Forgetting to square the components before adding in the magnitude.

Check yourself

Try these without looking back, then compare with the answers on the next page.

- Find the area of a trapezium with parallel sides 6 and 10 and height 4.
- Find the volume of a cylinder with radius 4 and height 5, in terms of π .
- A sector has radius 6 and angle 120° . Find its perimeter.
- Find $\sin 45^\circ \times \cos 45^\circ$.
- A ship sails on a bearing of 135° . What is the bearing back to the start?
- $\vec{PQ} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$. Find $|\vec{PQ}|$.

Answers to the self-check

1. $\frac{1}{2}(6 + 10) \times 4 = 32$.
2. $\pi \times 16 \times 5 = 80\pi$.
3. Arc = $\frac{1}{3} \times 12\pi = 4\pi$. Perimeter = $4\pi + 12$.
4. $\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{2}$.
5. $135 + 180 = 315^\circ$.
6. $\sqrt{9 + 1} = \sqrt{10}$.

Next step

Heurisco's *ESAT Maths 1: Area, Volume, Trigonometry and Vectors Practice Set* has twenty original exam-style questions on this lesson, in order of difficulty, with a full worked solution for each.