

ESAT

Maths 1

Applied reasoning and puzzles

Free lesson notes

Covers

Applies M3, M4, M5 and M7

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

Some of the most memorable ESAT questions do not name a topic at all. Two trains pass each other. A tank is filled by two taps while a third drains it. The hour and minute hands of a clock make a right angle. Nothing here needs more than GCSE mathematics, but each problem asks you to turn a situation into an equation, or to find a clever shortcut, and then to check that your answer makes sense.

This lesson teaches the handful of ideas behind almost all such questions: **relative speed**, **average speed**, **rates of work**, **clock angles**, **mixtures and ages**, and **systematic counting**. Each section gives the idea, a diagram, two or three fully worked examples, and the mistakes to avoid.

How to use it. Work through each example on paper before reading the solution. Then try the self-check, and finally the practice set. Throughout, follow the same habit the test rewards: *draw the situation, define your unknown, write one equation, solve it, then ask whether the answer is sensible.*

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1 Speed, distance and time

The three quantities are linked by

$$\text{distance} = \text{speed} \times \text{time}, \quad \text{speed} = \frac{\text{distance}}{\text{time}}, \quad \text{time} = \frac{\text{distance}}{\text{speed}}.$$

Keep units consistent. 1 m/s = 3.6 km/h, and 30 minutes is 0.5 hours. If a problem mixes metres with kilometres or seconds with hours, convert *before* you calculate.

Average speed is *total distance divided by total time*. It is not the mean of the speeds unless the times are equal. A journey out at 30 km/h and back at 60 km/h does not average 45 km/h, because you spend longer travelling at the slower speed.

Worked example: average speed

A car drives 60 km at 30 km/h and returns the same 60 km at 60 km/h. Find the average speed for the whole journey.

Time out = $\frac{60}{30} = 2$ h. Time back = $\frac{60}{60} = 1$ h. Total distance = 120 km, total time = 3 h.

Average speed = $\frac{120}{3} = 40$ km/h.

Worked example: a river

A boat travels at 8 km/h in still water on a river flowing at 2 km/h. It goes 12 km upstream and 12 km back. How long does the whole trip take?

Upstream the speed is $8 - 2 = 6$ km/h, so the time is $\frac{12}{6} = 2$ h. Downstream the speed is $8 + 2 = 10$ km/h, so the time is $\frac{12}{10} = 1.2$ h. Total = 3.2 h.

(The current slows the boat for longer than it speeds it up, so the trip takes more than the 3 hours it would take on still water.)

Common mistakes

- Averaging the two speeds. Always divide total distance by total time.
- Mixing units, such as metres and kilometres, or minutes and hours, in one equation.
- Forgetting that a current changes the speed in both directions, by opposite amounts.

2 Relative speed: trains and meeting problems

When two objects move on the same line, their *closing speed* decides how quickly the gap between them shrinks:

- moving **towards each other**: closing speed = $v_1 + v_2$;
- moving in the **same direction**, the faster one catching the slower: closing speed = $v_1 - v_2$.

The *gap to close* is the real insight for trains. Two trains have *completely passed* each other when the back of each has cleared the front of the other, so the gap to close is the sum of the two lengths. When a train passes a tunnel, bridge or platform, the distance is its own length plus the length of the structure.



Worked example: trains passing

Train A is 150 m long and travels at 20 m/s. Train B is 100 m long and travels at 30 m/s on the parallel track in the opposite direction. How long do they take to pass completely?

Closing speed = $20 + 30 = 50$ m/s. The gap to close is $150 + 100 = 250$ m.

$$\text{Time} = \frac{250}{50} = 5 \text{ s.}$$

Worked example: overtaking

Train A (200 m, 30 m/s) overtakes train B (150 m, 20 m/s) travelling the same way. How long does the overtake take, from the front of A reaching the back of B until the back of A clears the front of B?

Closing speed = $30 - 20 = 10$ m/s. Gap to close = $200 + 150 = 350$ m. Time = 35 s.

Worked example: a meeting problem

Two towns are 300 km apart. A train leaves each at the same moment and they travel towards each other at 70 and 80 km/h. A bird starts at one train and flies between the two at 120 km/h until the trains meet. How far does the bird fly?

Do not try to add up the bird's zig-zags. Find how long it flies: the trains close at 150 km/h, so they meet after $\frac{300}{150} = 2$ h. The bird flies for 2 h at 120 km/h, so it covers 240 km.

Common mistakes

- Using the length of only one train. Two trains need the sum of both lengths; a train and a tunnel need the sum of the train and the tunnel.
- Adding speeds when the objects move the same way, or subtracting when they move towards each other.
- Starting the clock at the wrong moment: "completely passed" ends when the *backs* clear, not the fronts.

3 Rates of work

If a tap fills a tank in 6 hours, it fills $\frac{1}{6}$ of the tank each hour. Rates *add* when the workers (or taps) work together, and a drain has a *negative* rate:

$$\text{combined rate} = \text{rate}_1 + \text{rate}_2 - \text{rate of drain}, \quad \text{time} = \frac{1}{\text{combined rate}}.$$

Always convert times into rates (jobs per hour) first. Adding the *times* is never correct.

Worked example: two taps

Tap A fills a tank in 6 hours and tap B in 3 hours. How long do they take together?

Rates: $\frac{1}{6} + \frac{1}{3} = \frac{1}{2}$ of the tank per hour, so the time is 2 hours.

Worked example: with a drain

Tap A fills a tank in 6 h, tap B in 12 h, and a drain empties a full tank in 18 h. All three are open. How long does it take to fill the empty tank?

$$\text{Net rate} = \frac{1}{6} + \frac{1}{12} - \frac{1}{18} = \frac{6+3-2}{36} = \frac{7}{36} \text{ per hour. Time} = \frac{36}{7} \text{ h} = 5\frac{1}{7} \text{ h.}$$

Worked example: walking up an escalator

You can walk up a stopped escalator in 60 s. The moving escalator alone carries you up in 40 s. How long does it take if you walk up the moving escalator?

$$\text{Your rates (staircases per second) add: } \frac{1}{60} + \frac{1}{40} = \frac{2+3}{120} = \frac{1}{24}. \text{ The time is 24 s.}$$

Common mistakes

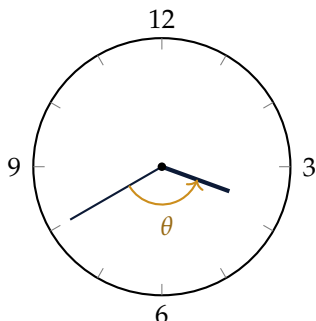
- Adding the times: “6 + 3 = 9 hours” for two taps. Together they must be *faster* than either alone.
- Forgetting to subtract a drain or an opposing rate.
- Giving the answer as a decimal when the fraction is exact: $\frac{36}{7}$, not 5.1.

4 Clock angles

A clock face is 360° . The minute hand turns 360° in 60 minutes, i.e. 6° per minute. The hour hand turns 360° in 12 hours, i.e. 30° per hour or 0.5° per minute. At h hours and m minutes (measured clockwise from 12):

$$\text{hour hand} = 30h + 0.5m, \quad \text{minute hand} = 6m, \quad \text{angle between} = |30h - 5.5m| \text{ (or } 360^\circ \text{ minus that).}$$

Take the *smaller* angle unless told otherwise. The hour hand does not sit exactly on the hour mark except on the hour: at 3:40 it is $\frac{2}{3}$ of the way from 3 to 4.



at 3:40:

$$\text{hour hand} = 30(3) + 0.5(40) = 110^\circ$$

$$\text{minute hand} = 6(40) = 240^\circ$$

$$\theta = 240 - 110 = 130^\circ$$

Worked example: the angle at 4:20

Find the angle between the hands at 4:20.

Hour hand: $30(4) + 0.5(20) = 130^\circ$. Minute hand: $6(20) = 120^\circ$. The angle is 10° .

(A common wrong answer is 0° : the hour hand is a little past the 4, not on it.)

Worked example: when are the hands opposite?

Between 2 and 3 o'clock, after how many minutes past 2 are the hands first exactly opposite?

At m minutes past 2: hour hand = $60 + 0.5m$, minute hand = $6m$. Opposite means a difference of 180° : $6m - (60 + 0.5m) = 180$, so $5.5m = 240$ and $m = \frac{480}{11} = 43\frac{7}{11}$ minutes.

Worked example: how many times?

How many times do the hands overlap between 12:00 and the next 12:00?

The minute hand gains 5.5° per minute on the hour hand, and it needs to gain 360° to catch it again, so overlaps

happen every $\frac{360}{5.5} = \frac{720}{11}$ minutes. In 720 minutes that is 11 overlaps (the last one coinciding with the next 12:00).

Common mistakes

- Treating the hour hand as fixed on the hour mark.
- Forgetting to take the smaller angle: an answer above 180° should be subtracted from 360° .
- Assuming the hands overlap 12 times in 12 hours: it is 11.

5 Mixtures, ages and unit-rate thinking

Mixtures. Track the amount of the *pure* substance, which does not change when you add water: (amount of pure) = (volume) $\times$ (concentration). **Ages.** Define one unknown for the youngest person and write every other age in terms of it; remember that *everyone* gains the same number of years. **Unit rates.** Reduce to “per one” (per hour, per person, per tap) before scaling up.

Worked example: diluting a solution

10 litres of a 30% solution has x litres of pure water added to make a 20% solution. Find x .

The pure solute is $0.3 \times 10 = 3$ litres and stays the same: $\frac{3}{10+x} = 0.2$, so $10+x = 15$ and $x = 5$.

Worked example: ages

A father is 4 times as old as his son. In 10 years he will be 2.5 times as old. How old is the father now?
Let the son be s and the father $4s$. Then $4s + 10 = 2.5(s + 10)$, so $1.5s = 15$, $s = 10$. The father is 40.

Common mistakes

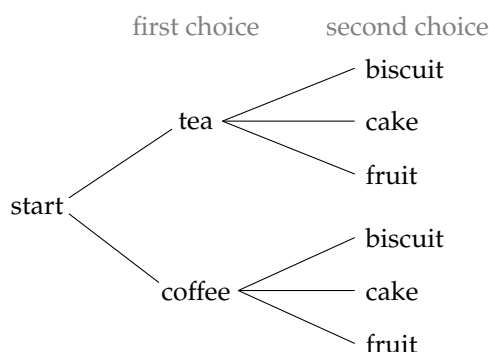
- Treating the *total* volume as pure when you dilute: adding water leaves the amount of pure substance unchanged.
- Applying the ratio of ages after the time has passed, instead of before.

6 Counting systematically

“How many...?” questions look easy and are full of traps. The cure is to *organise* your counting so that nothing is missed or counted twice. Four ideas cover almost every case: listing, the multiplication principle, arrangements, and choosing.

Listing. When the numbers are small, write every possibility in a fixed order (alphabetical, or smallest first). A systematic list is its own check. Always test a counting method on a small case you can list by hand before trusting it on a large one.

The multiplication principle. If one choice can be made in p ways and, *whichever way it is made*, a second choice can then be made in q ways, there are $p \times q$ ways to make both choices. A tree diagram shows why: each of the p branches splits into q further branches.



Here there are 2 drinks and 3 snacks, so $2 \times 3 = 6$ combinations. The same idea extends to any number of choices: multiply the number of options at each stage.

Worked example: how many three-digit numbers?

How many three-digit numbers have all three digits odd?

The hundreds digit can be any of 1, 3, 5, 7, 9: 5 ways. The tens digit: 5 ways. The units digit: 5 ways. By the multiplication principle there are $5 \times 5 \times 5 = 125$ numbers. (Compare: for *all* three-digit numbers the first digit has 9 options because it cannot be 0, and the others have 10, giving $9 \times 10 \times 10 = 900$.)

Arrangements and factorials. In how many orders can 4 different books be placed on a shelf? There are 4 choices for the first place, then 3 books remain for the second place, then 2, then 1. By the multiplication principle the number of orders is $4 \times 3 \times 2 \times 1 = 24$. This product has a name and a symbol: the *factorial*. For a positive whole number n ,

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1,$$

read “ n factorial”. So $3! = 6$, $4! = 24$, $5! = 120$, and $n!$ is the number of ways to arrange n different things in a row. (By convention $0! = 1$.) If only some of the places are filled, stop the product early: the number of ways to award gold, silver and bronze to 3 of 8 runners is $8 \times 7 \times 6 = 336$.

Worked example: keeping two items together

In how many ways can 5 people stand in a line if A and B must stand next to each other?

Treat A and B as a single block. Then there are 4 things to arrange (the block and the other three people): $4! = 24$ orders. Inside the block, A and B can be in either order: 2 ways. Total $24 \times 2 = 48$. (Without the restriction there would be $5! = 120$.)

Choosing: when order does not matter. Arrangements care about order. Often we only care about *which* items are chosen, not the order: a team of 3 chosen from 5 people is the same team however the names are listed. To count such selections, start by counting ordered selections and then correct for the repeats.

Worked reasoning. Choose a team of 3 from 5 people A, B, C, D, E . If order mattered there would be $5 \times 4 \times 3 = 60$ ordered selections. But each team, such as $\{A, B, C\}$, appears in $3! = 6$ different orders ($ABC, ACB, BAC, BCA, CAB, CBA$), and all six are the same team. So the number of teams is $\frac{60}{6} = 10$.

In general, choosing k items from n (order irrelevant) gives

$$\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

read “ n choose k ” (also written nC_k). The numerator is the k factors you get by choosing in order; dividing by $k!$ removes the repeats. Two useful facts: $\binom{n}{k} = \binom{n}{n-k}$ (choosing the k people to include is the same as choosing the $n-k$ to leave out), and $\binom{n}{1} = n$, $\binom{n}{0} = 1$.

Worked example: handshakes: why $n(n-1)/2$

Eight people each shake hands once with each other person. How many handshakes?

A handshake is a *pair* of people, and order does not matter, so we are choosing 2 people from 8: $\binom{8}{2} = \frac{8 \times 7}{2!} = 28$.

Direct reasoning agrees: each of the 8 people shakes hands with the other 7, giving $8 \times 7 = 56$ handshakes counted from each person’s point of view, but every handshake has been counted twice, so there are $56 \div 2 = 28$. In general

n people give $\frac{n(n-1)}{2}$ handshakes.

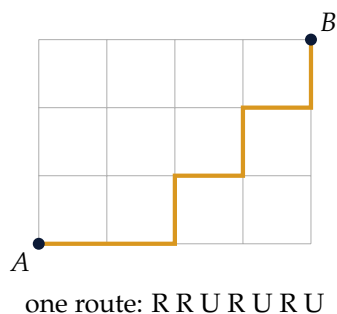
Worked example: choosing a committee

A committee of 3 is chosen from 4 women and 5 men. How many committees contain at least one woman?

Count all committees and subtract the bad ones. All committees: $\binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$. Committees with no

women (all men): $\binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$. So $84 - 10 = 74$ committees have at least one woman. (“At least one” is almost always easiest by counting the complement.)

Routes on a grid. To walk from corner A to the opposite corner B of a grid that is a squares across and b squares up, moving only right (R) or up (U), you must take exactly a steps right and b steps up, $a+b$ steps in all. A route is decided by *which* of the $a+b$ steps are the right ones, so there are $\binom{a+b}{a}$ routes.



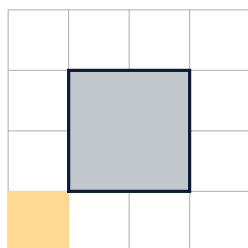
Worked example: routes on a grid

How many shortest routes are there from A to B on the 4×3 grid above?

Each route has 4 R steps and 3 U steps, 7 steps in total. Choose which 4 of the 7 steps are R: $\binom{7}{4} = \frac{7 \times 6 \times 5 \times 4}{4!} =$

35. (Equivalently $\binom{7}{3} = 35$.)

Counting squares in a grid. In an $n \times n$ grid of unit squares, the number of squares of side s is $(n - s + 1)^2$, because the top-left corner of such a square can be placed in $n - s + 1$ columns and $n - s + 1$ rows. Add over all sizes.



side 1: $4 \times 4 = 16$ squares

side 2: $3 \times 3 = 9$ squares

side 3: $2 \times 2 = 4$ squares

side 4: $1 \times 1 = 1$ square

Worked example: squares in a grid

How many squares of all sizes are there in a 4×4 grid?

Side 1: $4^2 = 16$; side 2: $3^2 = 9$; side 3: $2^2 = 4$; side 4: $1^2 = 1$. Total $16 + 9 + 4 + 1 = 30$.

Common mistakes

- Counting each handshake or each team twice: the pair $\{A, B\}$ is the same as $\{B, A\}$, so divide by the number of orders.
- Using arrangements $(n!)$ when order does not matter, or choosing $\binom{n}{k}$ when it does.
- Counting only the unit squares in a grid.
- Allowing a leading zero in a digit problem: 014 is not a three-digit number.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. A train 120 m long passes a pole in 8 s. How fast is it travelling, and how long does it take to cross a 480 m bridge completely?
2. Two cyclists start 36 km apart and ride towards each other at 12 and 6 km/h. When do they meet?
3. Find the angle between the hands of a clock at 2:30.
4. A pipe fills a pool in 4 h and a second in 12 h. How long do both take together?
5. How many handshakes are there among 10 people? Explain your method.
6. In how many ways can 4 different books be arranged on a shelf, and in how many ways can you choose 2 of the 4 to lend out?
7. A runner jogs 5 km at 10 km/h then 5 km at 5 km/h. What is the average speed?

Answers to the self-check

1. Speed = $\frac{120}{8} = 15$ m/s. Distance for the bridge = $120 + 480 = 600$ m, so the time is 40 s.
2. Closing speed 18 km/h, so $\frac{36}{18} = 2$ hours.
3. Hour = $30(2) + 15 = 75^\circ$; minute = 180° . The angle is 105° .
4. $\frac{1}{4} + \frac{1}{12} = \frac{1}{3}$, so 3 h.
5. Choose 2 people from 10 with order irrelevant: $\frac{10 \times 9}{2} = 45$ (each of the 10 people shakes 9 hands, and every handshake is counted twice).
6. Arrangements: $4! = 24$. Choices: $\binom{4}{2} = \frac{4 \times 3}{2} = 6$.
7. Time $0.5 + 1 = 1.5$ h for 10 km, so $\frac{10}{1.5} = 6\frac{2}{3}$ km/h.

Next step

Heurisco's *ESAT Maths 1: Applied Reasoning and Puzzles Practice Set* has twenty original questions in this style, from friendly to demanding, with a full worked solution for each.