

ESAT

Maths 2

Quadratics and polynomials

Free lesson notes

Covers	MM1.3, MM1.4 and MM1.6
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Quadratics are the first curves you meet, and they are everywhere in Maths 2: a ball thrown in the air, the largest area you can fence, the point where a line just touches a curve, the roots of a cubic. This lesson builds the whole subject from the ground up, using pictures to explain *why* each method works rather than only how.

We start with what a quadratic's graph looks like and where its special points are. We then *complete the square* (derived from a picture of squares and strips), and use the same idea once, in general, to *derive the quadratic formula* and the *discriminant*. Next come the links between a quadratic's roots and its coefficients, and what happens when a line meets a parabola (the idea of tangency). The second half moves from quadratics to polynomials of any degree: long division explained step by step, and the factor and remainder theorems, which turn division questions into simple substitutions. The last section collects hidden quadratics and modelling problems.

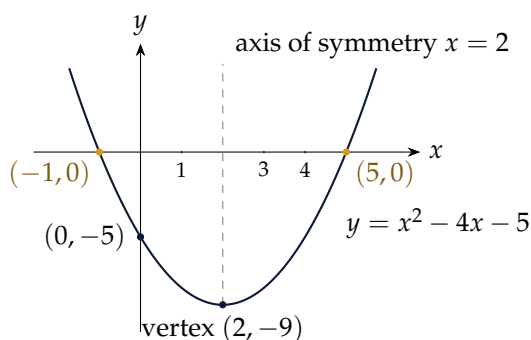
How to use it. Cover each worked example and attempt it before reading on; every example has been verified by computer. Always sketch the graph: it turns many questions into a picture. Finish with the self-check and then the practice set. Specification codes: MM1.3 (quadratics, discriminant, completing the square), MM1.4 (simultaneous equations with one quadratic) and MM1.6 (polynomials, algebraic division, factor and remainder theorems).

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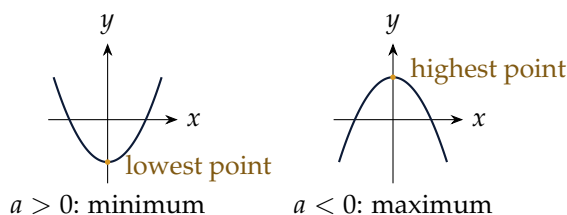
1 The anatomy of a quadratic

A **quadratic** is an expression $ax^2 + bx + c$ with $a \neq 0$. Its graph $y = ax^2 + bx + c$ is a smooth U-shaped curve called a **parabola**. Four features matter, and every method in this lesson is a way of finding one of them.

- The **y-intercept** is where $x = 0$, so it is $(0, c)$.
- The **roots** (or zeros) are the x -values where the curve meets the x -axis, i.e. the solutions of $ax^2 + bx + c = 0$. There can be two, one or none.
- The **axis of symmetry** is a vertical mirror line: the left half of the parabola is the reflection of the right half. If there are two roots, the axis is exactly half-way between them.
- The **vertex** is the turning point, which lies on the axis of symmetry. It is the lowest point if $a > 0$ and the highest if $a < 0$.



The sign of a decides the shape: if $a > 0$ the arms point up (a "smile" with a minimum) and if $a < 0$ they point down (a "frown" with a maximum). A larger $|a|$ makes the parabola narrower.



Worked example: reading the features from a factorised form

Find the roots, the axis of symmetry, the vertex and the y -intercept of $y = x^2 - 4x - 5$.
 Factorise: $x^2 - 4x - 5 = (x - 5)(x + 1)$, so the roots are $x = 5$ and $x = -1$. The axis is half-way between them: $x = \frac{5 + (-1)}{2} = 2$. The vertex is on the axis, so substitute $x = 2$: $y = 4 - 8 - 5 = -9$, giving $(2, -9)$. The y -intercept is $(0, -5)$. All of this is drawn in the figure above.

Worked example: a parabola from its roots and one point

A parabola crosses the x -axis at $x = 1$ and $x = 7$ and passes through $(0, 14)$. Find its equation and its lowest point.
 Any parabola with these roots has the form $y = a(x - 1)(x - 7)$ (the factor a can stretch it). At $x = 0$: $a(-1)(-7) = 7a = 14$, so $a = 2$ and $y = 2(x - 1)(x - 7) = 2x^2 - 16x + 14$. The axis is $x = \frac{1 + 7}{2} = 4$ and $y(4) = 2(3)(-3) = -18$: the lowest point is $(4, -18)$.

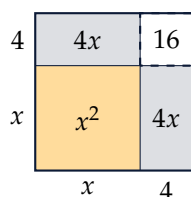
Common mistakes

- Forgetting the factor a when building an equation from its roots: $y = (x - 1)(x - 7)$ has the right roots but the wrong height.
- Finding the axis of symmetry but forgetting to substitute back to get the y -coordinate of the vertex.
- Thinking every parabola crosses the x -axis. The curve $y = x^2 + 1$ has no roots at all.

2 Completing the square

Factorising only works when the roots happen to be neat. **Completing the square** always works, and it also reveals the vertex. The idea is to rewrite $x^2 + bx$ as a perfect square plus a correction, and a picture shows how.

Take $x^2 + 8x$. Draw a square of side x (area x^2). Split the extra $8x$ into two strips, each x by 4 (area $4x$ each) and attach them to two sides of the square. Together the three pieces make a big square with a corner missing. The missing corner is $4 \times 4 = 16$, so adding 16 completes a square of side $x + 4$.



The amber square and the two grey strips make $x^2 + 4x + 4x = x^2 + 8x$.

Adding the dashed corner ($4 \times 4 = 16$) completes a big square of side $x + 4$.

So $x^2 + 8x = (x + 4)^2 - 16$.

In symbols: $x^2 + 8x + 16 = (x + 4)^2$, so $x^2 + 8x = (x + 4)^2 - 16$. We added 16 to complete the square, so we must take it away again. The general rule follows from the picture: for $x^2 + bx$ each strip has width $\frac{b}{2}$, the missing corner has area $\left(\frac{b}{2}\right)^2$, so

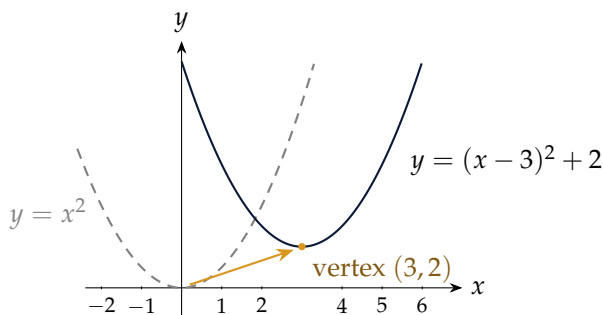
$$x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2.$$

Rule of thumb: halve the coefficient of x , put it in the bracket, then subtract its square.

When $a \neq 1$, first factor a out of the x^2 and x terms only, complete the square inside the bracket, then multiply out the bracket's correction: $ax^2 + bx + c = a \left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} \right] + c$.

Why it reveals the vertex. A square is never negative, so $a(x + p)^2 + q$ is smallest (if $a > 0$) when the bracket is zero, that

is when $x = -p$, and the smallest value is q . The vertex is $(-p, q)$. In graph terms, the curve $y = (x - 3)^2 + 2$ is the basic parabola $y = x^2$ moved 3 to the right and 2 up.



Worked example: completing the square and solving

Write $x^2 + 10x + 7$ in the form $(x + p)^2 + q$ and hence solve $x^2 + 10x + 7 = 0$ exactly.

Half of 10 is 5: $x^2 + 10x + 7 = (x + 5)^2 - 25 + 7 = (x + 5)^2 - 18$. Setting this equal to 0: $(x + 5)^2 = 18$, so $x + 5 = \pm\sqrt{18} = \pm 3\sqrt{2}$ and $x = -5 \pm 3\sqrt{2}$. The vertex is $(-5, -18)$.

Worked example: a leading coefficient that is not 1

Find the minimum value of $2x^2 - 12x + 7$ and the value of x where it occurs.

Factor 2 out of the first two terms: $2(x^2 - 6x) + 7 = 2[(x - 3)^2 - 9] + 7 = 2(x - 3)^2 - 18 + 7 = 2(x - 3)^2 - 11$. The square is smallest (0) at $x = 3$, so the minimum value is -11 , at $x = 3$.

Worked example: a maximum

Find the greatest value of $5 + 4x - x^2$.

Write it as $-(x^2 - 4x) + 5 = -[(x - 2)^2 - 4] + 5 = 9 - (x - 2)^2$. The term $(x - 2)^2$ is never negative, so $9 - (x - 2)^2 \leq 9$ with equality at $x = 2$. The greatest value is 9.

Common mistakes

- Using the wrong sign: $x^2 + 10x$ gives $(x + 5)^2$, not $(x - 5)^2$. Check by expanding.
- Forgetting to subtract the square you added: $x^2 + 10x + 7 = (x + 5)^2 - 18$, not $(x + 5)^2 + 7$.
- Factoring a out of all three terms, or forgetting to multiply the correction by a : in $2(x - 3)^2 - 11$ the -18 came from $2 \times (-9)$.
- Reading the vertex of $a(x + p)^2 + q$ as (p, q) . It is $(-p, q)$.

3 The quadratic formula and the discriminant

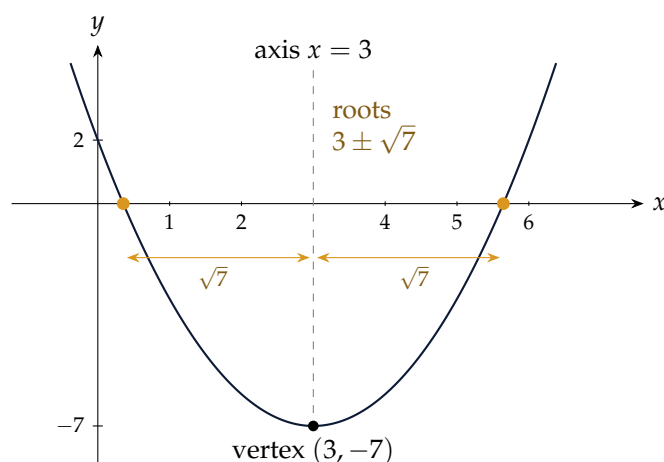
Complete the square once, on the general equation $ax^2 + bx + c = 0$, and every quadratic is solved at once. Divide by a , then complete the square:

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 \\ \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\ \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \pm \frac{\sqrt{b^2 - 4ac}}{2a} \end{aligned}$$

so that

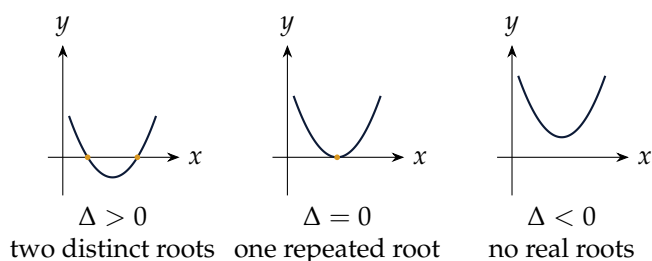
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The picture explains the structure of the formula. The first part, $-\frac{b}{2a}$, is the axis of symmetry (the vertex's x -coordinate). The roots are the same distance, $\frac{\sqrt{b^2 - 4ac}}{2a}$, either side of it.

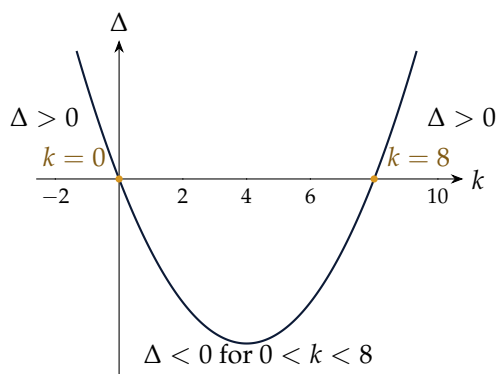


The discriminant. The expression under the square root, $\Delta = b^2 - 4ac$, decides how many real roots there are, because we can only take the square root of a number that is not negative.

- $\Delta > 0$: the square root is a positive number, giving *two distinct real roots* (the curve crosses the x -axis twice).
- $\Delta = 0$: the $\pm$ gives the same answer twice, so there is *one repeated root* (the vertex touches the x -axis).
- $\Delta < 0$: no real square root, so *no real roots* (the curve never reaches the x -axis).



Equations with an unknown constant. Many questions say “find k so that...”. Write the discriminant in terms of k , which is itself a quadratic expression in k , and use the sketch of Δ against k to decide where it is positive, zero or negative. For $x^2 + kx + 2k = 0$ we get $\Delta = k^2 - 8k = k(k - 8)$.



Worked example: using the formula

Solve $2x^2 - 5x - 1 = 0$.

Here $a = 2$, $b = -5$, $c = -1$, so $\Delta = (-5)^2 - 4(2)(-1) = 25 + 8 = 33$ and $x = \frac{5 \pm \sqrt{33}}{4}$. (The roots are about 2.69 and -0.19.)

Worked example: a condition on k

For which values of k does $x^2 + kx + 2k = 0$ have (a) equal roots, (b) no real roots?

$\Delta = k^2 - 4(1)(2k) = k^2 - 8k = k(k - 8)$. (a) Equal roots need $\Delta = 0$: $k = 0$ or $k = 8$. (b) No real roots need $\Delta < 0$. The graph of $\Delta = k^2 - 8k$ is a smile that is below the axis between its roots, so $0 < k < 8$.

Worked example: always positive

Show that $2x^2 - 4x + 5 > 0$ for every real x .

Complete the square: $2(x^2 - 2x) + 5 = 2(x - 1)^2 - 2 + 5 = 2(x - 1)^2 + 3 \geq 3 > 0$. (Equivalently $\Delta = 16 - 40 = -24 < 0$ and $a > 0$, so the graph is a smile that never reaches the x -axis.)

Common mistakes

- Writing $\Delta = b^2 - 4ac$ with a sign error when c is negative: $-4ac$ is then *positive*.
- Dividing only part of the numerator by $2a$ in the formula. The whole of $-b \pm \sqrt{\Delta}$ is divided.
- Using $\Delta \geq 0$ for "two distinct roots". Two distinct roots need $\Delta > 0$; $\Delta \geq 0$ means real roots, possibly equal.
- Forgetting that in $kx^2 + \dots$ the coefficient a is k , so you must also require $k \neq 0$ for the equation to be a quadratic at all.

4 The roots and the coefficients

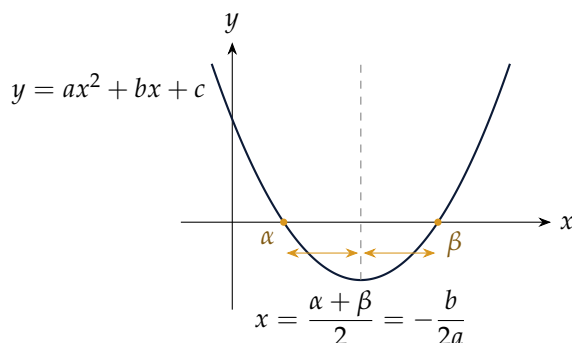
If a quadratic has roots α and β , then it factorises as $a(x - \alpha)(x - \beta)$. Expand, taking $a = 1$ to start with:

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta.$$

Comparing with $x^2 + \frac{b}{a}x + \frac{c}{a}$ (the equation $ax^2 + bx + c = 0$ divided by a) gives

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

The sum rule is the picture's statement that the axis $x = -\frac{b}{2a}$ is half-way between the roots: $\frac{\alpha + \beta}{2} = -\frac{b}{2a}$.



These two facts let you work with roots *without solving the equation*. The trick is to express what you want using only $\alpha + \beta$ and $\alpha\beta$:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta,$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta},$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta).$$

To build a new equation with new roots α' and β' , find their sum S and product P ; the equation is $x^2 - Sx + P = 0$.

Worked example: symmetric expressions

The roots of $x^2 - 5x + 2 = 0$ are α and β . Find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$.
 $\alpha + \beta = 5$ and $\alpha\beta = 2$. Then $\alpha^2 + \beta^2 = 25 - 2(2) = 21$ and $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{5}{2}$.

Worked example: an equation with new roots

The roots of $x^2 - 3x + 1 = 0$ are α and β . Find the equation with roots 2α and 2β .
 $\alpha + \beta = 3$, $\alpha\beta = 1$. New sum $= 2\alpha + 2\beta = 6$; new product $= 4\alpha\beta = 4$. The equation is $x^2 - 6x + 4 = 0$.

Worked example: a relationship between the roots

One root of $x^2 - 6x + k = 0$ is five times the other. Find k .
 Call the roots r and $5r$. Their sum is $6r = 6$, so $r = 1$ and the roots are 1 and 5. Their product is $k = 5$. Check: $x^2 - 6x + 5 = (x - 1)(x - 5)$.

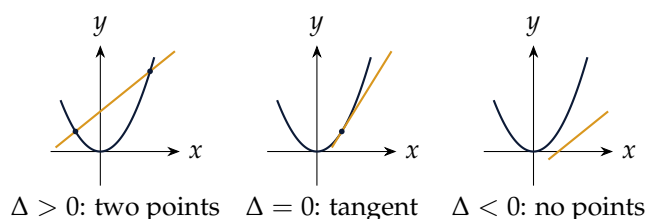
Common mistakes

- Mixing up the signs: the sum of the roots is $-\frac{b}{a}$ (with a minus), the product is $+\frac{c}{a}$.
- Forgetting to divide by a when $a \neq 1$: for $2x^2 - 5x + 3 = 0$ the sum is $\frac{5}{2}$ and the product $\frac{3}{2}$.
- Writing $\alpha^2 + \beta^2 = (\alpha + \beta)^2$. The cross term $2\alpha\beta$ must be subtracted.

5 When a line meets a parabola

To find where a line meets a curve, solve their equations simultaneously. For a line and a parabola, *substitute* the line's y into the parabola's equation: you get a quadratic in x . Its roots are the x -coordinates of the meeting points, and you then find each y from the (simpler) line equation. Because we are solving a quadratic, its discriminant tells you the geometry:

- $\Delta > 0$: the line cuts the curve at two points (a *secant*).
- $\Delta = 0$: the two meeting points merge into one, so the line *touches* the curve (a *tangent*).
- $\Delta < 0$: the line misses the curve completely.

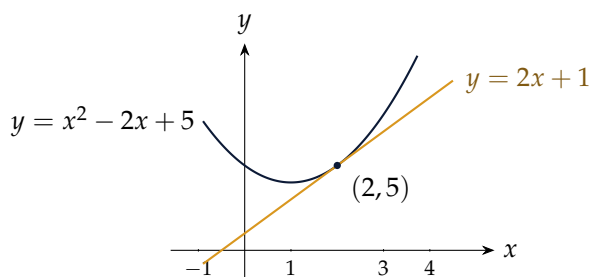


Worked example: two intersection points

The line $y = 2x - 1$ meets $y = x^2 - 2x + 2$ at points A and B . Find the length AB .
 Substitute: $x^2 - 2x + 2 = 2x - 1$, so $x^2 - 4x + 3 = 0$ and $(x - 1)(x - 3) = 0$. The points are $A(1, 1)$ and $B(3, 5)$ (using $y = 2x - 1$). Length $= \sqrt{(3 - 1)^2 + (5 - 1)^2} = \sqrt{20} = 2\sqrt{5}$.

Worked example: a tangent

Find k so that $y = 2x + k$ is a tangent to $y = x^2 - 2x + 5$, and the point of contact.
 Substitute: $x^2 - 2x + 5 = 2x + k$, so $x^2 - 4x + (5 - k) = 0$. A tangent means one repeated root: $\Delta = 16 - 4(5 - k) = 4k - 4 = 0$, so $k = 1$. The repeated root is $x = -\frac{b}{2a} = 2$ and $y = 2(2) + 1 = 5$: the contact point is $(2, 5)$.



Common mistakes

- Finding the x -values and stopping. Substitute back into the *line* to get the matching y -values.
- Substituting into the curve when both are curves of degree 2 or higher: substituting into the line is simpler and avoids false pairs.
- Not rearranging to " $= 0$ " before computing the discriminant. Move everything to one side first.

6 Polynomials and long division

A **polynomial** is a sum of terms $a_n x^n$ where each power n is a whole number: $x^3 - x^2 - 5x + 11$, $4x^5 - 2$, 7 . The highest power is its **degree** (so a quadratic has degree 2, a cubic degree 3) and the numbers multiplying the powers are its **coefficients**. You add and subtract polynomials by collecting like terms, and multiply by expanding as you did in Maths 1.

Dividing. When we say $17 \div 5 = 3$ remainder 2 we mean $17 = 5 \times 3 + 2$, with a remainder smaller than the divisor. Polynomials work the same way: dividing $f(x)$ by $(x - a)$ gives a **quotient** $q(x)$ and a constant **remainder** r with

$$f(x) = (x - a)q(x) + r.$$

(The remainder must have smaller degree than the divisor. For a divisor of degree 1 that means a number.)

Before the standard method, here is why it works: multiplying $(x - 2)(x^2 + x - 3)$ gives six terms, and when they are added and the remainder 5 is included we get $x^3 - x^2 - 5x + 11$.

	x^2	$+x$	-3
x	x^3	x^2	$-3x$
-2	$-2x^2$	$-2x$	$+6$

Add the six pieces:
 $x^3 + x^2 - 2x^2 - 3x - 2x + 6$
 $= x^3 - x^2 - 5x + 6$.
 So $(x - 2)(x^2 + x - 3) + 5$
 $= x^3 - x^2 - 5x + 11$.

Long division undoes this multiplication one term at a time. At each step: (i) divide the *leading term* of what remains by the leading term of the divisor to get the next term of the quotient; (ii) multiply the whole divisor by that term and write the result underneath; (iii) subtract, taking care with signs, and bring down the next term. Stop when what is left has lower degree than the divisor.

$x - 2$	$ \begin{array}{r} x^2 \quad +x \quad -3 \\ x^3 \quad -x^2 \quad -5x \quad +11 \\ \underline{x^3 \quad -2x^2} \\ x^2 \quad -5x \\ \underline{x^2 \quad -2x} \\ -3x \quad +11 \\ \underline{-3x \quad +6} \\ 5 \end{array} $	quotient: each step divides by x $x^2 \times (x - 2)$, subtract bring down $-5x$ $x \times (x - 2)$, subtract bring down $+11$ $-3 \times (x - 2)$, subtract remainder (degree below 1: stop)
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Reading the figure: $x^3 \div x = x^2$; $x^2(x - 2) = x^3 - 2x^2$ is subtracted, leaving $x^2 - 5x$; $x^2 \div x = x$, and $x(x - 2) = x^2 - 2x$ is subtracted, leaving $-3x + 11$; $-3x \div x = -3$, and $-3(x - 2) = -3x + 6$ is subtracted, leaving the remainder 5. So $x^3 - x^2 - 5x + 11 = (x - 2)(x^2 + x - 3) + 5$.

Missing powers. If a power is absent, write it with coefficient 0 so the columns line up: $x^3 - 8 = x^3 + 0x^2 + 0x - 8$.

Worked example: dividing by a linear factor

Divide $6x^3 + x^2 - 4x + 3$ by $(2x - 1)$.
 $6x^3 \div 2x = 3x^2$. Subtract $3x^2(2x - 1) = 6x^3 - 3x^2$: leaves $4x^2 - 4x$. Then $4x^2 \div 2x = 2x$. Subtract $2x(2x - 1) = 4x^2 - 2x$: leaves $-2x + 3$. Then $-2x \div 2x = -1$. Subtract $-1(2x - 1) = -2x + 1$: leaves 2. So the quotient is $3x^2 + 2x - 1$ and the remainder is 2.

Worked example: a missing term

Divide $x^3 - 8$ by $(x - 2)$.
 Write $x^3 + 0x^2 + 0x - 8$. Then $x^3 \div x = x^2$; subtract $x^3 - 2x^2$ to leave $2x^2 + 0x$; $2x^2 \div x = 2x$; subtract $2x^2 - 4x$ to leave $4x - 8$; $4x \div x = 4$; subtract $4x - 8$ to leave 0. So $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$ with remainder 0.

Common mistakes

- Subtracting a negative incorrectly: $(-2x^2) - (+x^2)$ and $-(-3x)$ are the usual traps. Write the line to subtract in full and change every sign.
- Forgetting to include a zero coefficient for a missing power, which misaligns the columns.
- Dividing the *whole* remaining expression, rather than only its leading term, by the leading term of the divisor.

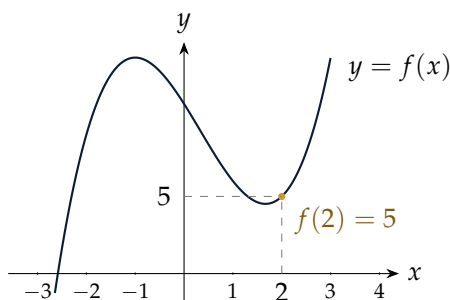
7 The factor and remainder theorems

Long division is slow. The **remainder theorem** avoids it when all you want is the remainder. Start from the division identity $f(x) = (x - a)q(x) + r$ and put $x = a$:

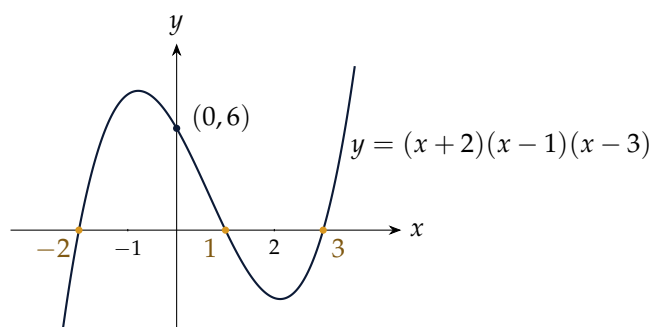
$$f(a) = (a - a)q(a) + r = 0 + r = r.$$

Remainder theorem. When a polynomial $f(x)$ is divided by $(x - a)$, the remainder is $f(a)$. More generally, when divided by $(px - q)$ the remainder is $f\left(\frac{q}{p}\right)$ (the value of x that makes the divisor zero).

Factor theorem. If the remainder is zero then $(x - a)$ divides exactly, so $(x - a)$ is a *factor*. Therefore $(x - a)$ is a factor of $f(x)$ if and only if $f(a) = 0$. In graph terms, the point $(a, 0)$ is on the curve, so a is a root. The graph below shows $f(x) = x^3 - 5x^2 + 11x - 6$: its height at $x = 2$ is $f(2) = 5$, which is exactly the remainder found by long division in the last section.



Factorising a cubic. (1) Look for a root by trying small values $x = \pm 1, \pm 2, \dots$ (for a polynomial with integer coefficients and leading coefficient 1, any integer root divides the constant term). (2) When $f(a) = 0$, divide by $(x - a)$ to get a quadratic. (3) Factorise or solve the quadratic. The graph of a cubic with three roots looks like this:



Worked example: the remainder without dividing

Find the remainder when $f(x) = 2x^3 - 5x^2 + x - 4$ is divided by (a) $x - 3$, (b) $2x + 1$.

(a) $f(3) = 54 - 45 + 3 - 4 = 8$. (b) The divisor is zero at $x = -\frac{1}{2}$: $f\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{8}\right) - 5\left(\frac{1}{4}\right) - \frac{1}{2} - 4 = -\frac{1}{4} - \frac{5}{4} - \frac{1}{2} - 4 = -6$.

Worked example: factorising a cubic

Factorise $x^3 - 3x^2 - 6x + 8$ completely.

Try $x = 1$: $1 - 3 - 6 + 8 = 0$, so $(x - 1)$ is a factor. Divide: $x^3 - 3x^2 - 6x + 8 = (x - 1)(x^2 - 2x - 8)$. The quadratic factorises as $(x - 4)(x + 2)$. So the cubic is $(x - 1)(x - 4)(x + 2)$, with roots 1, 4, -2.

Worked example: finding unknown coefficients

$f(x) = x^3 + ax^2 + bx - 6$ has $(x - 1)$ as a factor and leaves remainder 4 when divided by $(x + 1)$. Find a and b .

Factor: $f(1) = 1 + a + b - 6 = 0$, so $a + b = 5$. Remainder: $f(-1) = -1 + a - b - 6 = 4$, so $a - b = 11$. Adding: $2a = 16$, $a = 8$, $b = -3$.

Worked example: dividing by a quadratic

A polynomial $p(x)$ leaves remainder 2 when divided by $(x - 1)$ and remainder 5 when divided by $(x - 2)$. Find the remainder when it is divided by $(x - 1)(x - 2)$.

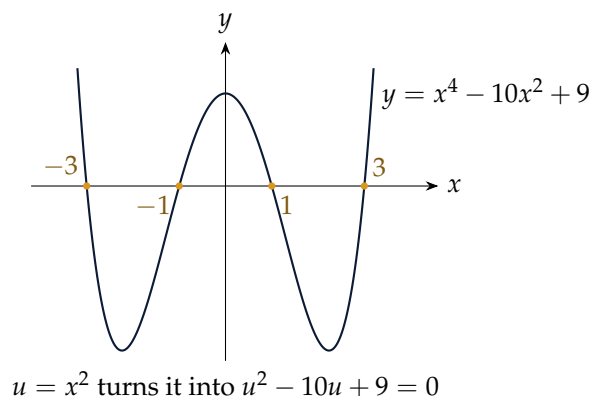
The divisor has degree 2, so the remainder has degree at most 1: write it $ax + b$. Then $p(x) = (x - 1)(x - 2)Q(x) + ax + b$. Put $x = 1$: $a + b = p(1) = 2$. Put $x = 2$: $2a + b = p(2) = 5$. Subtract: $a = 3$, $b = -1$. The remainder is $3x - 1$.

Common mistakes

- Using $f(-a)$ instead of $f(a)$: the factor $(x - 3)$ goes with $f(3)$, and $(x + 2)$ goes with $f(-2)$.
- For a divisor like $2x + 1$, using $f(1)$ or $f(2)$ instead of $f\left(-\frac{1}{2}\right)$: solve "divisor = 0" for x .
- Stopping once one root is found. A cubic can have three roots, so divide and then factorise the quadratic that remains.

8 Hidden quadratics and quadratic models

Hidden quadratics. Some equations are not quadratics in x but are quadratics in a function of x . If x^4 appears with x^2 (and no odd powers), put $u = x^2$. Solve for u , then go back: each positive u gives *two* values of x , a negative u gives none. The same idea works with $u = \sqrt{x}$, $u = 2^x$ or $u = x + \frac{1}{x}$.



Worked example: an equation in x^2

Solve $x^4 - 10x^2 + 9 = 0$.

Put $u = x^2$: $u^2 - 10u + 9 = 0$, so $(u - 1)(u - 9) = 0$ and $u = 1$ or 9 . Then $x^2 = 1$ gives $x = \pm 1$ and $x^2 = 9$ gives $x = \pm 3$. Four roots: -3, -1, 1, 3 (the figure above).

Worked example: the largest fenced area

A farmer has 60 m of fencing to enclose a rectangular field with a long straight wall as one side (so no fence is needed there). Find the greatest area.

Let each side perpendicular to the wall be x . The side parallel to the wall is $60 - 2x$, so $A = x(60 - 2x) = 60x - 2x^2$. Complete the square: $A = -2(x^2 - 30x) = -2[(x - 15)^2 - 225] = 450 - 2(x - 15)^2$. The maximum is 450 m^2 , at $x = 15$ (then the other side is 30 m). By symmetry the maximum is half-way between the two roots $x = 0$ and $x = 30$ of $A = 0$.

Worked example: least value of a sum of squares

$x + y = 10$. Find the least possible value of $x^2 + y^2$.

Substitute $y = 10 - x$: $x^2 + (10 - x)^2 = 2x^2 - 20x + 100 = 2(x - 5)^2 + 50$. The least value is 50, when $x = y = 5$.

Exam tip

A quick order of attack for “quadratic” questions: (1) can it be factorised? (2) does it ask for a vertex, a maximum or a minimum? Then complete the square. (3) does it ask about the number of roots, or about a tangent? Then use the discriminant. (4) is it about roots without solving? Then use sum and product. The more careful sketch you draw, the fewer algebra mistakes you make.

Common mistakes

- Forgetting to square-root back: after $u = x^2 = 9$ there are two values of x , ± 3 .
- Accepting $u = x^2 = -4$. A square cannot be negative, so that value gives no real x .
- In optimisation questions, giving the value of x when the question asks for the maximum *area* (or the other way round).

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Write $x^2 - 8x + 3$ in the form $(x - p)^2 + q$.
2. The graph of $y = x^2 + bx + c$ crosses the x -axis at $x = -1$ and $x = 5$. Find b , c and the vertex.
3. For which k does $x^2 + kx + 9 = 0$ have equal roots?
4. The roots of $2x^2 - 6x + 1 = 0$ are α and β . Find $\alpha^2 + \beta^2$.
5. Find k so that $y = x + k$ is tangent to $y = x^2 + x + 3$.
6. Divide $x^3 - 2x^2 - 5x + 6$ by $(x - 3)$.
7. The remainder when $x^3 + kx - 4$ is divided by $(x - 2)$ is 10. Find k .
8. Solve $x^4 - 5x^2 + 4 = 0$.

Answers to the self-check

1. $(x - 4)^2 - 13$.
2. $y = (x + 1)(x - 5) = x^2 - 4x - 5$, so $b = -4$, $c = -5$. Axis $x = 2$, vertex $(2, -9)$.
3. $\Delta = k^2 - 36 = 0$, so $k = \pm 6$.
4. $\alpha + \beta = 3$, $\alpha\beta = \frac{1}{2}$, so $9 - 1 = 8$.
5. $x^2 + x + 3 = x + k$ gives $x^2 + (3 - k) = 0$; $\Delta = -4(3 - k) = 0$ so $k = 3$.
6. Quotient $x^2 + x - 2$, remainder 0 (since $27 - 18 - 15 + 6 = 0$).
7. $8 + 2k - 4 = 10$, so $k = 3$.
8. $u = x^2$: $(u - 1)(u - 4) = 0$, so $x = \pm 1, \pm 2$.

Next step

Heurisco's *ESAT Maths 2: Quadratics and Polynomials Practice Set* has twenty original exam-style questions on this lesson, with diagrams where they help, in order of difficulty, with a full worked solution for each.