

ESAT

Maths 2

Indices, surds, inequalities and functions

Free lesson notes

Covers	MM1.1, MM1.2, MM1.5, MM1.7
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

This lesson extends Maths 1 in four directions: indices with any rational power (including reduced quadratics in $x^{1/3}$), surds and nested surds, linear and quadratic inequalities, and the language of functions: domain, range, one-to-one and many-to-one mappings, composite and inverse functions.

How to use it. Try each worked example first. Then do the self-check, then the practice set. For inequalities, a quick sketch of the quadratic's graph removes most sign errors.

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1 Rational indices and hidden quadratics

All the Maths 1 index laws hold for any rational power: $a^{m/n} = (\sqrt[n]{a})^m$. A recurring Maths 2 pattern is an equation that is quadratic in a power of x . Substitute $u = x^{1/3}$, $u = x^2$ or $u = 3^x$, solve the quadratic in u , then go back.

Worked example: a quadratic in $x^{1/3}$

Solve $x^{2/3} - 3x^{1/3} + 2 = 0$.
Let $u = x^{1/3}$: $u^2 - 3u + 2 = 0$, so $u = 1$ or $u = 2$.
Then $x = u^3$: $x = 1$ or $x = 8$.

Worked example: a quadratic in 3^x

Solve $3^{2x} - 10 \times 3^x + 9 = 0$.
Let $u = 3^x$: $u^2 - 10u + 9 = 0$, so $u = 1$ or $u = 9$. Then $x = 0$ or $x = 2$.

Common mistakes

- Stopping at the values of u : always convert back to x .
- Forgetting a rejected solution: $u = 3^x$ can never be negative or zero.

2 Surds

Beyond Maths 1, expect nested surds and fractions with two surds. Use $(\sqrt{a} \pm \sqrt{b})^2 = a + b \pm 2\sqrt{ab}$ to recognise a perfect square under a root: $7 + 4\sqrt{3} = (2 + \sqrt{3})^2$ because $4 + 3 = 7$ and $2 \times 2 \times \sqrt{3} = 4\sqrt{3}$. To rationalise a denominator $\sqrt{a} - \sqrt{b}$ multiply top and bottom by $\sqrt{a} + \sqrt{b}$ to get $a - b$.

Worked example: a nested surd

Simplify $\sqrt{7 + 4\sqrt{3}} - \sqrt{7 - 4\sqrt{3}}$.
 $7 \pm 4\sqrt{3} = (2 \pm \sqrt{3})^2$, so the expression is $(2 + \sqrt{3}) - (2 - \sqrt{3}) = 2\sqrt{3}$ (both brackets are positive).

Worked example: rationalising

Write $\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} - \sqrt{3}}$ in the form $a + \sqrt{b}$.
Multiply by $\frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}$: numerator $(\sqrt{5} + \sqrt{3})^2 = 8 + 2\sqrt{15}$, denominator $5 - 3 = 2$.
Result: $4 + \sqrt{15}$.

Common mistakes

- Taking $\sqrt{(2 - \sqrt{3})^2} = 2 - \sqrt{3}$ only when $2 - \sqrt{3} > 0$. Check the sign of what is under the root.
- Rationalising with the same sign (squaring the denominator) instead of the conjugate.

3 Inequalities

Linear inequalities are solved like equations, reversing the sign when multiplying or dividing by a negative. For a quadratic inequality $ax^2 + bx + c \geq 0$:

1. Find the roots (set the quadratic equal to 0).
2. Sketch the parabola (with $a > 0$, a $\cup$ shape).
3. Read off the x -values where the graph is above (> 0) or below (< 0) the axis.

Above the axis means *outside* the roots; below means *between* them. For an inequality like $(x - 1)(x + 3) > 12$, move everything to one side *first*.

Worked example: outside the roots

Solve $(x - 1)(x + 3) > 12$.
 $x^2 + 2x - 3 > 12$, so $x^2 + 2x - 15 > 0$ and $(x + 5)(x - 3) > 0$.
 The graph is above the axis outside the roots: $x < -5$ or $x > 3$.

Worked example: true for all x

Find the integer values of k for which $x^2 + kx + 4 > 0$ for all real x .
 The graph never meets the axis, so $\Delta = k^2 - 16 < 0$, giving $-4 < k < 4$. The integers are $-3, \dots, 3$; the largest is 3.

Common mistakes

- Writing $-5 < x < 3$ (between the roots) for a > 0 inequality. Above the axis is *outside*.
- Dividing both sides by x or by $(x - a)$: the sign of x is unknown. Factorise instead.

4 Functions: domain, range and type

A function f maps each input in its *domain* to exactly one output; the set of outputs is the *range*.

- **One-to-one**: different inputs give different outputs (every horizontal line meets the graph at most once). Example: $f(x) = x^3 + x$.
- **Many-to-one**: some outputs come from more than one input. Example: $f(x) = x^2$ (since $f(2) = f(-2)$).

The *largest domain* excludes values that make a denominator zero or put a negative under a square root. To find a range, complete the square or use the graph.

Worked example: domain

Find the largest real domain of $f(x) = \sqrt{6 - x - x^2}$.
 We need $6 - x - x^2 \geq 0$, i.e. $x^2 + x - 6 \leq 0$, so $(x + 3)(x - 2) \leq 0$ and $-3 \leq x \leq 2$.

Worked example: range

Find the range of $f(x) = x^2 - 4x + 7$ for all real x .
 $f(x) = (x - 2)^2 + 3 \geq 3$, with 3 attained at $x = 2$. The range is $f(x) \geq 3$.

Common mistakes

- Writing the range as $f(x) > 3$ when 3 is actually reached.
- Calling $x^3 - x$ one-to-one: it has turning points, so horizontal lines meet it up to three times.

5 Composite and inverse functions

$fg(x) = f(g(x))$: apply g first, then f . In general $fg \neq gf$. The inverse f^{-1} undoes f : write $y = f(x)$, make x the subject, then swap names. Only one-to-one functions have inverses over their whole domain, and $f(f^{-1}(x)) = x$.

Worked example: composites

$f(x) = x + 2$ and $g(x) = x^2 - 1$. Find $gf(x) - fg(x)$.
 $gf(x) = (x + 2)^2 - 1 = x^2 + 4x + 3$ and $fg(x) = x^2 - 1 + 2 = x^2 + 1$.
 $gf(x) - fg(x) = 4x + 2$.

Worked example: inverse

$f(x) = 3x - 1$. Solve $ff(x) = f^{-1}(x)$.
 $ff(x) = 3(3x - 1) - 1 = 9x - 4$ and $f^{-1}(x) = \frac{x + 1}{3}$.
 $9x - 4 = \frac{x + 1}{3}$ gives $27x - 12 = x + 1$, so $x = \frac{1}{2}$.

Common mistakes

- Applying the functions in the wrong order: $fg(x)$ means g first.
- Confusing $f^{-1}(x)$ with $\frac{1}{f(x)}$.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Solve $x^2 - 4x - 5 < 0$.
2. Solve $x^4 - 10x^2 + 9 = 0$.
3. Simplify $\frac{1}{3 - \sqrt{5}}$.
4. Find the range of $f(x) = 5 - x^2$.
5. $f(x) = 2x + 1$, $g(x) = x^2$. Find $fg(3)$ and $gf(3)$.
6. Find $f^{-1}(x)$ for $f(x) = \frac{x - 4}{3}$.

Answers to the self-check

1. $(x - 5)(x + 1) < 0$, so $-1 < x < 5$.
2. $u = x^2$: $u = 1, 9$, so $x = \pm 1, \pm 3$.
3. $\frac{3 + \sqrt{5}}{9 - 5} = \frac{3 + \sqrt{5}}{4}$.
4. $f(x) \leq 5$.
5. $fg(3) = 2 \times 9 + 1 = 19$. $gf(3) = 7^2 = 49$.
6. $y = \frac{x-4}{3} \Rightarrow x = 3y + 4$, so $f^{-1}(x) = 3x + 4$.

Next step

Heurisco's *ESAT Maths 2: Indices, Surds, Inequalities and Functions Practice Set* has twenty original exam-style questions on this lesson, in order of difficulty, with a full worked solution for each.