

ESAT

Maths 2

Sequences and series

Free lesson notes

Covers

MM2.1 to MM2.4

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

A *sequence* is a list of numbers in a definite order, and a *series* is what you get when you add the terms of a sequence together. They appear whenever a pattern repeats: seats in a theatre, savings, a bouncing ball, the digits of a recurring decimal. This lesson teaches the four ideas the specification asks for, and, importantly, *why* each formula is true, so that you can rebuild it if you forget it.

We start with how to describe a sequence (by a formula for the n th term or by a recurrence relation) and the notation for adding terms up. Then we study the two important families: **arithmetic** sequences (add the same amount each time) and **geometric** sequences (multiply by the same factor each time), including the surprising fact that infinitely many terms can add up to a finite number. We finish with counting, factorials and the binomial coefficient $\binom{n}{r}$ (“ n choose r ”), which give the binomial expansion of $(1+x)^n$. Specification codes: MM2.1 (n th term and recurrence), MM2.2 (arithmetic series), MM2.3 (geometric series), MM2.4 (binomial expansion).

How to use it. Work every example yourself first. When a formula is derived, follow the derivation once, then close the page and try to reproduce it: that is the best preparation for questions that vary the standard set-up.

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1 Describing sequences

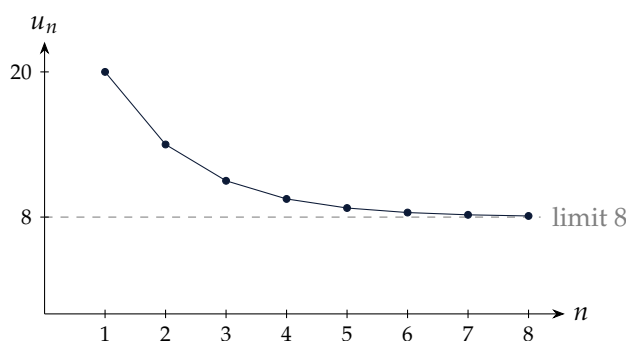
A sequence is written $u_1, u_2, u_3, \dots$, where u_n is the n th term. There are two ways to define one.

By a formula for the n th term. For example $u_n = 3n - 2$ gives $1, 4, 7, 10, \dots$ (put $n = 1, 2, 3, \dots$). This lets you jump straight to any term: $u_{50} = 148$.

By a recurrence relation. This gives each term from the one before, together with a starting value. For example $u_{n+1} = 2u_n - 3$ with $u_1 = 5$ means “to get the next term, double the current term and subtract 3”. You must calculate the terms in order: $u_2 = 2(5) - 3 = 7$, $u_3 = 2(7) - 3 = 11$, $u_4 = 19$. The subscript $n+1$ means “the term after u_n ”, so it is *not* the same as $u_n + 1$.

Sigma notation. The symbol Σ (capital Greek letter sigma) means “add up”. The expression $\sum_{r=1}^4 (3r - 2)$ means: substitute $r = 1, 2, 3, 4$ into $3r - 2$ and add the results, giving $1 + 4 + 7 + 10 = 22$. The number underneath is where the counting starts and the number above is where it stops.

Limits. Some recurrences settle down towards a value. If the terms approach a limit L , then for large n both u_{n+1} and u_n are (almost) equal to L , so L must satisfy $L = f(L)$ where $u_{n+1} = f(u_n)$. This lets you find the limit without calculating many terms. The graph below shows $u_{n+1} = \frac{1}{2}u_n + 4$ starting from $u_1 = 20$: the terms are $20, 14, 11, 9.5, \dots$ and creep towards 8.



Worked example: a recurrence

$u_{n+1} = 2u_n - 3$ and $u_1 = 5$. Find u_4 .
 $u_2 = 2(5) - 3 = 7$, $u_3 = 2(7) - 3 = 11$, $u_4 = 2(11) - 3 = 19$.

Worked example: a limit

The sequence $u_{n+1} = \frac{1}{2}u_n + 4$ approaches a limit L . Find L .
 At the limit u_{n+1} and u_n both equal L , so $L = \frac{1}{2}L + 4$, giving $\frac{1}{2}L = 4$ and $L = 8$. The starting value does not matter.
 (A recurrence $u_{n+1} = ku_n + c$ only settles when $|k| < 1$; if $|k| > 1$ the terms run away from the fixed point.)

Worked example: using sigma notation

Evaluate $\sum_{r=1}^5 (r^2 + 1)$.

Substitute $r = 1, \dots, 5$: $(1 + 1) + (4 + 1) + (9 + 1) + (16 + 1) + (25 + 1) = 2 + 5 + 10 + 17 + 26 = 60$.

Common mistakes

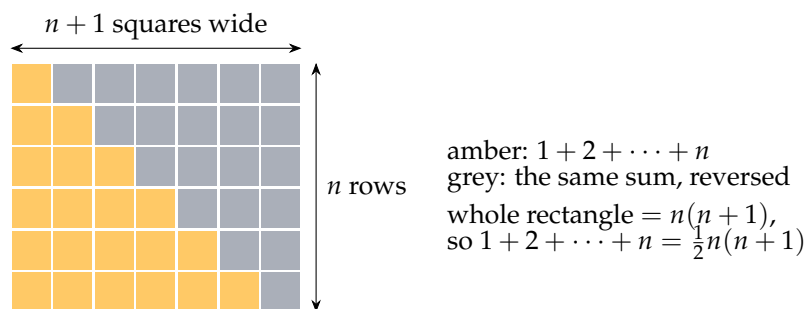
- Reading u_{n+1} as $u_n + 1$. It means the *next term*, not one more than the current term.
- Assuming every recurrence settles to a limit. Only when the multiplier has size less than 1 does it converge.
- Starting the sum at the wrong place: $\sum_{r=0}^4$ has five terms, $\sum_{r=1}^4$ has four.

2 Arithmetic sequences and series

In an **arithmetic sequence** each term is the previous one plus the same *common difference* d . Starting from the first term a , the terms are $a, a + d, a + 2d, \dots$. Notice that the n th term has had d added $n - 1$ times (the first term has had none), so

$$u_n = a + (n - 1)d.$$

The sum of an arithmetic series. Let S_n be the sum of the first n terms. The famous trick is to add the series to itself written backwards. Take $1 + 2 + 3 + \dots + n$. Written forwards and backwards, each pair of numbers in the same position adds to $n + 1$, and there are n pairs, so twice the sum is $n(n + 1)$. The diagram shows the same idea with squares: two staircases fit together to make a rectangle of $n(n + 1)$ squares.



Exactly the same argument works for any arithmetic series: every pair (first and last, second and second-last, and so on) has the same total, $a + l$, where $l = a + (n - 1)d$ is the last term. Therefore

$$S_n = \frac{n}{2} (a + l) = \frac{n}{2} (2a + (n - 1)d).$$

In particular, the sum of the first n positive integers is $\frac{n(n + 1)}{2}$.

Finding a term from the sum. S_n includes the first n terms and S_{n-1} includes the first $n - 1$. Their difference is exactly the n th term: $u_n = S_n - S_{n-1}$.

Pairs of terms. If you are told two terms (say the 5th and the 12th), write each as $a + (k - 1)d$ and solve the two equations simultaneously: subtracting them gives d immediately.

Worked example: from two terms

The 5th term of an arithmetic sequence is 17 and the 12th is 38. Find the sum of the first 20 terms.
The 12th term is 7 steps beyond the 5th, so $7d = 38 - 17 = 21$ and $d = 3$. Then $a = 17 - 4 \times 3 = 5$ (the 5th term is $a + 4d$). $S_{20} = \frac{20}{2}(2 \times 5 + 19 \times 3) = 10 \times 67 = 670$.

Worked example: how many terms?

An arithmetic series has first term 7, last term 52 and sum 472. How many terms does it have?
 $S_n = \frac{n}{2}(a + l)$ gives $472 = \frac{n}{2} \times 59$, so $n = \frac{944}{59} = 16$. (The common difference is then $\frac{52 - 7}{15} = 3$.)

Worked example: from the sum formula

The sum of the first n terms of a sequence is $S_n = 3n^2 + 2n$. Find the 10th term and show the sequence is arithmetic.
 $u_{10} = S_{10} - S_9 = (300 + 20) - (243 + 18) = 320 - 261 = 59$. In general $u_n = S_n - S_{n-1} = 3n^2 + 2n - (3(n-1)^2 + 2(n-1)) = 6n - 1$, which increases by 6 each time, so the sequence is arithmetic with $d = 6$.

Common mistakes

- Using $a + nd$ instead of $a + (n - 1)d$ for the n th term.
- Confusing S_n (the sum of n terms) with u_n (the n th term). The 10th term is $S_{10} - S_9$, not S_{10} .
- Forgetting the $\frac{1}{2}$ in $S_n = \frac{n}{2}(a + l)$: the pairs trick counts every term twice.

3 Geometric sequences and series

In a **geometric sequence** each term is the previous one multiplied by the same *common ratio* r : $a, ar, ar^2, ar^3, \dots$. The n th term has been multiplied by r a total of $n - 1$ times, so $u_n = ar^{n-1}$. To find r , divide any term by the one before it. If you are given two terms, divide them to cancel a : for example if $ar = 6$ and $ar^4 = 162$ then $r^3 = 27$.

The sum of a geometric series. Let $S_n = a + ar + ar^2 + \dots + ar^{n-1}$. Multiply the whole equation by r : $rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$. Now subtract. Almost every term cancels, leaving $S_n - rS_n = a - ar^n$. Factorising, $S_n(1 - r) = a(1 - r^n)$, so

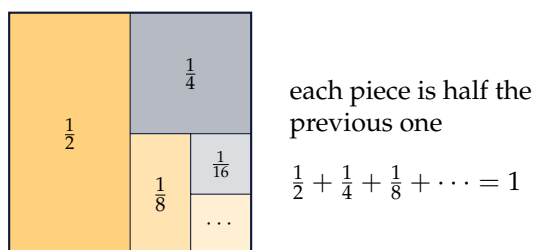
$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1} \quad (r \neq 1).$$

Use whichever form has a positive denominator.

Infinite series. Suppose $|r| < 1$, for instance $r = \frac{1}{2}$. Then r^n shrinks towards 0 as n grows ($\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$), so $a(1 - r^n)$ approaches a and the sum approaches

$$S_\infty = \frac{a}{1 - r} \quad (|r| < 1).$$

If $|r| \geq 1$ the terms do not shrink and the sum does not settle, so the formula must not be used. The picture shows the idea: a square of area 1 cut repeatedly in half has pieces $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ which together fill the square.



Three consecutive terms. If p, q, s are consecutive terms of a geometric sequence then $\frac{q}{p} = \frac{s}{q}$, so $q^2 = ps$ and $q = \pm\sqrt{ps}$ (the *geometric mean*). (For an *arithmetic* sequence the middle term is instead the ordinary mean $\frac{p+s}{2}$.)

Recurring decimals. A recurring decimal is an infinite geometric series in disguise.

Worked example: a finite sum

Find the sum of the first 8 terms of 3, 6, 12, 24, ...

$$a = 3 \text{ and } r = 2. S_8 = \frac{3(2^8 - 1)}{2 - 1} = 3 \times 255 = 765.$$

Worked example: two terms given

The 2nd term of a geometric sequence is 6 and the 5th term is 162. Find the 8th term.

$ar = 6$ and $ar^4 = 162$. Dividing: $r^3 = 27$, so $r = 3$ and $a = 2$. The 8th term is $ar^7 = 2 \times 3^7 = 4374$.

Worked example: recurring decimal as a fraction

Express $0.\dot{3}\dot{6} = 0.363636\dots$ as a fraction.

$$0.363636\dots = 0.36 + 0.0036 + 0.000036 + \dots \text{ is geometric with } a = 0.36 \text{ and } r = 0.01. S_\infty = \frac{0.36}{1 - 0.01} = \frac{36}{99} = \frac{4}{11}.$$

Common mistakes

- Using the infinite-sum formula when $|r| \geq 1$.
- Powers: the sum of 8 terms uses r^8 (the power in the *formula*), while the 8th *term* uses r^7 .
- Calling a sequence geometric after checking only one ratio. Check that the ratio is the same each time.

4 Factorials and the binomial coefficient

Before the binomial expansion we need a little counting, because the numbers in the expansion count something.

Arrangements and factorials. In how many different orders can 4 different books be put on a shelf? There are 4 choices for the first position, then 3 books remain for the second, then 2, then 1. Multiplying: $4 \times 3 \times 2 \times 1 = 24$. This product is called *4 factorial*, written $4!$. In general

$$n! = n \times (n - 1) \times \dots \times 2 \times 1,$$

the number of ways to arrange n different objects in a row. So $3! = 6$, $5! = 120$, and by convention $0! = 1$.

Choosing, with order ignored. Now suppose we want to *choose* a group of r items from n without caring about order (a committee, a hand of cards). Choosing in order gives $n \times (n - 1) \times \dots \times (n - r + 1)$ possibilities (r factors). But each group has been counted once for every way of ordering its r members, which is $r!$ ways, so we divide by $r!$. The result is the **binomial coefficient**

$$\binom{n}{r} = \frac{n(n - 1) \dots (n - r + 1)}{r!} = \frac{n!}{r!(n - r)!},$$

read " n choose r ". For example, the number of ways to choose 3 people from 5 is $\binom{5}{3} = \frac{5 \times 4 \times 3}{3 \times 2 \times 1} = 10$.

Useful facts. Choosing the r items to *include* is the same as choosing the $n - r$ to *leave out*, so $\binom{n}{r} = \binom{n}{n - r}$ (e.g.

$$\binom{10}{8} = \binom{10}{2} = 45). \text{ Also } \binom{n}{0} = \binom{n}{n} = 1 \text{ and } \binom{n}{1} = n.$$

Pascal's triangle. Writing the coefficients row by row gives a triangle in which each entry is the sum of the two above it. The rule is $\binom{n}{r} = \binom{n - 1}{r - 1} + \binom{n - 1}{r}$: to choose r items from n , look at one particular item; either it is chosen (then pick $r - 1$ more from the other $n - 1$) or it is not (pick all r from the other $n - 1$).

$n = 0$				1			
$n = 1$			1	1			
$n = 2$			1	2	1		
$n = 3$			1	3	3	1	
$n = 4$		1	4	6	4	1	
$n = 5$	1	5	10	10	5	1	
$n = 6$	1	6	15	20	15	6	1

each number is the sum of the two above it: $4 + 6 = 10$

Worked example: evaluating coefficients

Evaluate $\binom{9}{3}$ and find n if $\binom{n}{2} = 28$.

$\binom{9}{3} = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$. For the second part, $\binom{n}{2} = \frac{n(n-1)}{2} = 28$ gives $n(n-1) = 56 = 8 \times 7$, so $n = 8$.

Worked example: handshakes

At a meeting, each of 10 people shakes hands once with each other person. How many handshakes?

A handshake is a pair of people, and order does not matter, so we choose 2 from 10: $\binom{10}{2} = \frac{10 \times 9}{2} = 45$.

Common mistakes

- Dividing by $r!$ but forgetting the factors: $\binom{n}{r}$ has r factors on top, not n .
- Using $\binom{n}{r}$ when order matters (then use the ordered count $n(n-1) \cdots (n-r+1)$).
- Writing $0! = 0$: it is 1.

5 The binomial expansion

To see where the binomial coefficients appear, expand $(1+x)^3 = (1+x)(1+x)(1+x)$. Multiplying out, each bracket contributes either a 1 or an x , giving $2^3 = 8$ products. Group them by how many x s they contain:

number of x s	0	1	2	3
products	$1 \cdot 1 \cdot 1$	$x \cdot 1 \cdot 1, 1 \cdot x \cdot 1, 1 \cdot 1 \cdot x$	$xx \cdot 1, x \cdot 1x, 1xx$	xxx
count	1	3	3	1

so $(1+x)^3 = 1 + 3x + 3x^2 + x^3$. The coefficient of x^k is the number of ways to choose which k of the n brackets supply an x , namely $\binom{n}{k}$. Therefore, for a positive integer n ,

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \cdots + x^n.$$

These are exactly the rows of Pascal's triangle: row 5 is 1, 5, 10, 10, 5, 1.

More general brackets. In $(a+bx)^n$ each choice of r brackets supplying the bx term contributes $a^{n-r}(bx)^r$, so the term in x^r is $\binom{n}{r}a^{n-r}b^r x^r$. Careful with signs and powers: in $(1-2x)^5$ we have $b = -2$, so the signs alternate and the x^3 term has $(-2)^3 = -8$.

Products of brackets. To find a coefficient in, say, $(1+x)(1-2x)^5$, expand the power far enough, then collect the ways to make the required power of x from each factor.

Approximations. For small x the powers $x^2, x^3, \dots$ are tiny, so the first few terms are a good approximation: $(1+x)^n \approx 1 + nx + \binom{n}{2}x^2$.

Worked example: a coefficient

Find the coefficient of x^3 in $(1+2x)^5$ and of x^2 in $(2-x)^6$.

x^3 in $(1+2x)^5$: $\binom{5}{3}(2x)^3 = 10 \times 8x^3 = 80x^3$. For $(2-x)^6$: the term in x^2 is $\binom{6}{2}2^4(-x)^2 = 15 \times 16x^2 = 240x^2$ (the power of 2 is $6-2=4$).

Worked example: a product

Find the coefficient of x^2 in $(1+x)(1-2x)^5$.

$(1-2x)^5 = 1 + 5(-2x) + 10(-2x)^2 + \cdots = 1 - 10x + 40x^2 - \cdots$. Now multiply by $(1+x)$: the x^2 terms come

from $1 \times 40x^2$ and $x \times (-10x)$, giving $40 - 10 = 30$.

Worked example: an approximation

Use the first three terms of the expansion to estimate 1.02^5 .

$$1.02^5 = (1 + 0.02)^5 \approx 1 + 5(0.02) + 10(0.02)^2 = 1 + 0.1 + 0.004 = 1.104.$$

Common mistakes

- Forgetting to raise the coefficient b to a power: in $(1 + 2x)^5$ the x^3 term has $2^3 = 8$ in it.
- Losing a sign: $(1 - 2x)^5$ alternates, and every odd power of x has a negative coefficient.
- Using the wrong power of a in $(a + bx)^n$: the powers of a and bx add up to n .

6 Strategy

- **Identify the type.** Constant *difference* means arithmetic; constant *ratio* means geometric; a bracket raised to a power calls for the binomial expansion.
- **Two pieces of information, two unknowns.** Terms given as “the p th term is ... and the q th term is ...” give two equations for a and d (or a and r). Subtract (arithmetic) or divide (geometric) to eliminate a .
- **“First exceeds”.** For questions like “when does the total first exceed 1000?” write the sum, then test values or solve an inequality, and check the term either side of your answer.
- **Sanity checks.** A sum to infinity must be bigger than the first term (for $0 < r < 1$). The x^0 term of a binomial expansion is the constant a^n .

Exam tip

If you forget a formula, derive it on the page: the “write it backwards and add” trick for arithmetic series, and the “multiply by r and subtract” trick for geometric series, take only two lines.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. $u_{n+1} = 3u_n - 2$ with $u_1 = 2$. Find u_4 .
2. Find the sum of the first 30 terms of 4, 7, 10, ...
3. The first term of a geometric series is 12 and the sum to infinity is 18. Find r .
4. Find the sum of the first 6 terms of 2, 6, 18, ...
5. Evaluate $\binom{9}{3}$ and explain in one sentence what it counts.
6. Find n if $\binom{n}{2} = 28$.
7. Find the coefficient of x^3 in $(1 + 3x)^5$.
8. Write $0.\dot{7} = 0.7777 \dots$ as a fraction using a geometric series.

Answers to the self-check

1. $u_2 = 4, u_3 = 10, u_4 = 28$.
2. $d = 3: \frac{30}{2}(8 + 29 \times 3) = 15 \times 95 = 1425$.
3. $\frac{12}{1-r} = 18$, so $1-r = \frac{2}{3}$ and $r = \frac{1}{3}$.
4. $a = 2, r = 3: S_6 = \frac{2(3^6 - 1)}{2} = 728$.
5. $\frac{9 \times 8 \times 7}{3 \times 2 \times 1} = 84$: the number of ways to choose 3 items from 9 when order does not matter.
6. $\frac{n(n-1)}{2} = 28$, so $n(n-1) = 56$ and $n = 8$.
7. $\binom{5}{3} \times 3^3 = 10 \times 27 = 270$.
8. $a = 0.7, r = 0.1: \frac{0.7}{0.9} = \frac{7}{9}$.

Next step

Heurisco's *ESAT Maths 2: Sequences and Series Practice Set* has twenty original exam-style questions on this lesson, in order of difficulty, with a full worked solution for each.