

# ESAT

## *Maths 2*

### Trigonometry

Free lesson notes

Covers

**MM4.1 to MM4.6**

Format

**Lesson, worked examples, self-check**

Price

**Free**

## About this lesson

Trigonometry begins with a simple observation: in a right-angled triangle the *shape* (the angles) fixes the *ratios* of the sides. That gives a way to turn angles into lengths and lengths into angles, and it grows into the study of anything that repeats: waves, tides, rotating wheels, orbits.

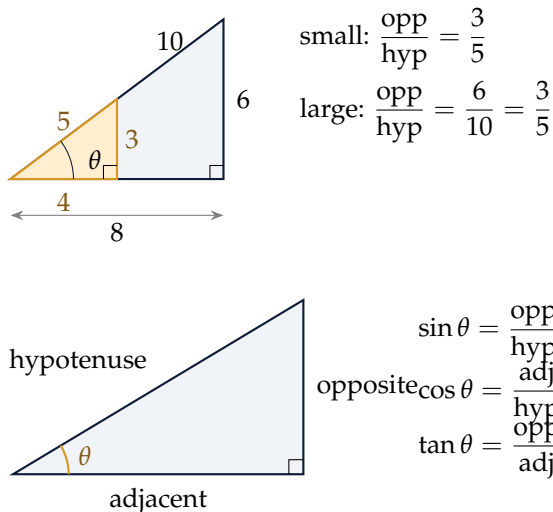
This lesson builds the subject in order. We start from right-angled triangles and the special angles whose values can be found exactly; extend to any triangle with the sine rule, the cosine rule and the area formula (including the “ambiguous case”); measure angles in radians and use them for arcs, sectors and segments; extend sine, cosine and tangent to *every* angle with the unit circle; sketch and transform their graphs; meet the two basic identities; and finally solve trigonometric equations within a given interval. A last section collects applied and intuitive ideas.

**How to use it.** Cover each worked example and try it first. No calculator is allowed in ESAT, so every value you need is either one of the exact values in Section 1 and Section 4 or can be built from them: learn those cold. Specification codes: MM4.1 (sine and cosine rules, area, ambiguous case), MM4.2 (radians, arcs, sectors), MM4.3 (exact values), MM4.4 (graphs, symmetry, periodicity), MM4.5 ( $\tan = \sin / \cos$  and  $\sin^2 + \cos^2 = 1$ ) and MM4.6 (simple equations in an interval).

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## 1 Right-angled triangles and exact values

**Why ratios work.** Take any right-angled triangle with an angle  $\theta$ , and draw a larger or smaller triangle with the same angles. The two are *similar*: every side is multiplied by the same scale factor, so ratios of sides do not change. In the figure the small triangle has sides 3, 4, 5 and the large one 6, 8, 10; the ratio opposite / hypotenuse is  $\frac{3}{5}$  in both. A ratio that depends only on the angle deserves a name.



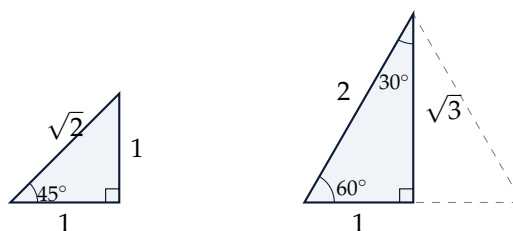
Label the sides from the point of view of the angle  $\theta$ : the **hypotenuse** is opposite the right angle (always the longest side), the **opposite** side faces  $\theta$ , and the **adjacent** side touches  $\theta$  (and is not the hypotenuse). Then

$$\sin \theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos \theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan \theta = \frac{\text{opp}}{\text{adj}}.$$

(The memory aid is SOH–CAH–TOA.) To find a side, choose the ratio that contains the side you want and the side you know; to find an angle, use the inverse ratio.

**Exact values from two special triangles.** Without a calculator we can still find  $\sin$ ,  $\cos$  and  $\tan$  of  $30^\circ$ ,  $45^\circ$  and  $60^\circ$ . Half of a square of side 1 is a right-angled triangle with angles  $45^\circ$ ,  $45^\circ$  and hypotenuse  $\sqrt{2}$  (Pythagoras). Half of an

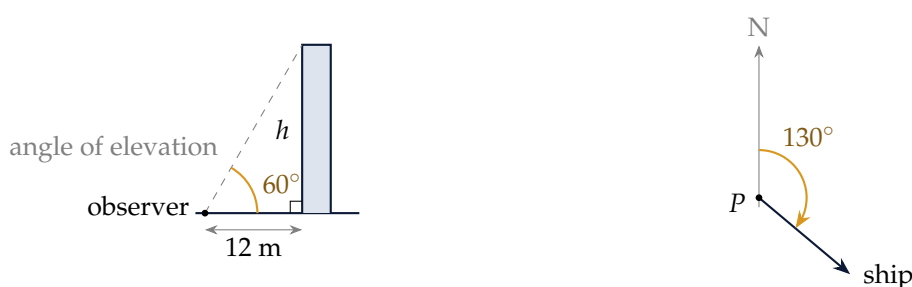
equilateral triangle of side 2 is a right-angled triangle with angles  $30^\circ, 60^\circ$ , hypotenuse 2, shortest side 1 and third side  $\sqrt{2^2 - 1^2} = \sqrt{3}$ . Reading ratios off these triangles:



| $\theta$      | $30^\circ$                                | $45^\circ$           | $60^\circ$           |
|---------------|-------------------------------------------|----------------------|----------------------|
| $\sin \theta$ | $\frac{1}{2}$                             | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ |
| $\cos \theta$ | $\frac{\sqrt{3}}{2}$                      | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$        |
| $\tan \theta$ | $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ | 1                    | $\sqrt{3}$           |

Notice the pattern in the sine row:  $\frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}$ , and that cosine is the same list backwards (because the other acute angle of the triangle is  $90^\circ - \theta$ :  $\sin \theta = \cos(90^\circ - \theta)$ ).

**Angles of elevation and bearings.** The *angle of elevation* of an object is the angle above the horizontal from the observer's eye to the object (the *angle of depression* is the angle below the horizontal). A *bearing* is the direction of travel measured *clockwise from north*, always written with three figures: east is  $090^\circ$ , south  $180^\circ$ , west  $270^\circ$ . The bearing back the way you came (the *back bearing*) differs by  $180^\circ$ .



#### Worked example: height from an angle of elevation

From a point 12 m from the foot of a vertical tower, the angle of elevation of the top is  $60^\circ$ . Find the height.

The known side is the adjacent (12) and the wanted side is the opposite ( $h$ ), so use tangent:  $\tan 60^\circ = \frac{h}{12}$ , giving  $h = 12\sqrt{3} \approx 20.8$  m.

#### Worked example: a ramp

A ramp 10 m long rises at  $30^\circ$  to the horizontal. How high does it rise, and what horizontal distance does it cover? Rise =  $10 \sin 30^\circ = 5$  m (opposite and hypotenuse). Horizontal distance =  $10 \cos 30^\circ = 5\sqrt{3} \approx 8.7$  m. Check with Pythagoras:  $5^2 + (5\sqrt{3})^2 = 25 + 75 = 100 = 10^2$ .

#### Worked example: a bearing

A boat sails 10 km on a bearing of  $120^\circ$ . How far east and how far south has it gone? What is its back bearing?

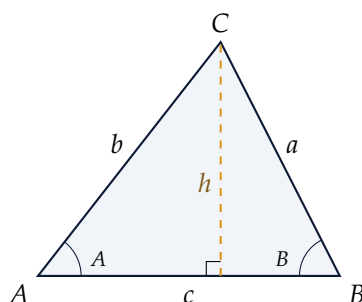
A bearing of  $120^\circ$  is  $30^\circ$  beyond east, towards south. East =  $10 \cos 30^\circ = 5\sqrt{3}$  km and south =  $10 \sin 30^\circ = 5$  km. The back bearing is  $120^\circ + 180^\circ = 300^\circ$ .

### Common mistakes

- Using the wrong side as “opposite”: it is opposite the angle you are using, so it changes if you switch to the other acute angle.
- Putting the hypotenuse in the numerator of sin or cos. The hypotenuse is the longest side, so  $\sin \theta$  and  $\cos \theta$  are never more than 1 in a right-angled triangle.
- Measuring a bearing anticlockwise, or giving it with fewer than three figures ( $45^\circ$  should be  $045^\circ$ ).

## 2 Non-right-angled triangles: the sine rule, cosine rule and area

For a triangle  $ABC$  we write  $a, b, c$  for the sides opposite  $A, B, C$ , so that the side  $a$  is *opposite* the angle  $A$ .

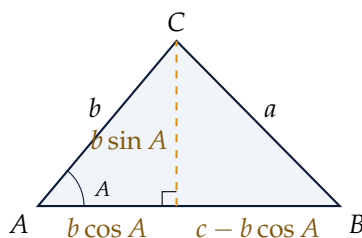


**The sine rule.** Drop a perpendicular of height  $h$  from  $C$  to  $AB$ . From the left-hand right-angled triangle,  $h = b \sin A$ ; from the right-hand one,  $h = a \sin B$ . So  $b \sin A = a \sin B$ , which rearranges to

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

The same argument works for any pair of sides. Use the sine rule when you know a side and its opposite angle (a “matching pair”) plus one more piece of information.

**The cosine rule.** Now drop the perpendicular from  $C$  and split  $AB$  into the piece  $b \cos A$  (next to  $A$ ) and the rest,  $c - b \cos A$ .



Pythagoras in the right-hand triangle gives  $a^2 = (b \sin A)^2 + (c - b \cos A)^2 = b^2 \sin^2 A + c^2 - 2bc \cos A + b^2 \cos^2 A$ . Since  $\sin^2 A + \cos^2 A = 1$  (Section 6),

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \Longleftrightarrow \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

When  $A = 90^\circ$  the last term vanishes and this is Pythagoras’ theorem. Use the cosine rule when you know two sides and the included angle (to find the third side), or all three sides (to find an angle). The sign matters: if the answer for  $\cos A$  is negative the angle is obtuse.

**Area.** The height of the triangle above is  $h = b \sin A$ , so the area is  $\frac{1}{2}ch = \frac{1}{2}bc \sin A$ . In general

$$\text{Area} = \frac{1}{2}ab \sin C,$$

the product of two sides and the *sine of the angle between them*, halved. **Which rule?** Matching side–angle pair: sine rule. Two sides and the angle between: cosine rule (then area formula). Three sides: cosine rule for an angle.

### Worked example: sine rule for a side

In triangle  $ABC$ ,  $A = 30^\circ$ ,  $B = 45^\circ$  and  $BC = 10$ . Find  $AC$ .

$BC = a$  is opposite  $A$  and  $AC = b$  is opposite  $B$ , so  $\frac{b}{\sin 45^\circ} = \frac{10}{\sin 30^\circ}$  and  $b = \frac{10 \cdot \frac{\sqrt{2}}{2}}{\frac{1}{2}} = 10\sqrt{2}$ .

### Worked example: cosine rule for a side

Two sides of a triangle are 5 and 8 with an included angle of  $60^\circ$ . Find the third side.

$a^2 = 5^2 + 8^2 - 2(5)(8) \cos 60^\circ = 25 + 64 - 40 = 49$ , so  $a = 7$ .

### Worked example: cosine rule for an angle

A triangle has sides 3, 5 and 7. Find its largest angle.

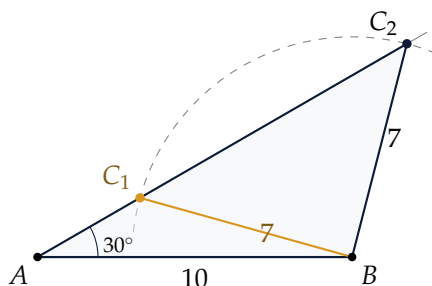
The largest angle is opposite the longest side.  $\cos \theta = \frac{3^2 + 5^2 - 7^2}{2(3)(5)} = \frac{-15}{30} = -\frac{1}{2}$ , so  $\theta = 120^\circ$  (obtuse, as the negative cosine warns).

### Worked example: area

Find the area of a triangle with sides 8 and 5 and an included angle of  $150^\circ$ .

$\frac{1}{2}(8)(5) \sin 150^\circ = 20 \times \frac{1}{2} = 10$ . (Sine of an obtuse angle is positive: see Section 4.)

**The ambiguous case.** Suppose you are given two sides and an angle that is *not* between them (SSA). The shape may not be fixed. In the figure,  $A = 30^\circ$ ,  $AB = 10$  and  $BC = 7$ : swing a compass of radius 7 about  $B$  and it meets the ray from  $A$  in *two* places,  $C_1$  and  $C_2$ , giving two different triangles.



The sine rule finds  $\sin C = \frac{10 \sin 30^\circ}{7} = \frac{5}{7}$ , and there are two angles between  $0^\circ$  and  $180^\circ$  with the same sine:  $C$  and  $180^\circ - C$ . The cosine rule shows both lengths of  $AC$  at once: it is a quadratic. With  $AC = b$ :  $7^2 = b^2 + 10^2 - 2(10)b \cos 30^\circ$ , so  $b^2 - 10\sqrt{3}b + 51 = 0$  and  $b = 5\sqrt{3} \pm 2\sqrt{6}$  (both positive, so both triangles exist). With the given angle  $A$  acute and the side opposite it  $a$  and the other known side  $c$ : if  $a < c \sin A$  there is no triangle; if  $a = c \sin A$  there is one (right-angled); if  $c \sin A < a < c$  there are two; if  $a \geq c$  there is only one.

### Worked example: an ambiguous triangle

In triangle  $ABC$ ,  $A = 60^\circ$ ,  $AB = 8$  and  $BC = 7$ . Find the possible lengths of  $AC$ .

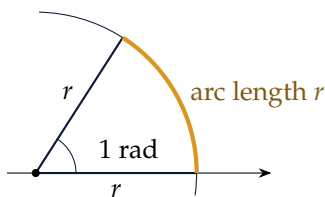
$7^2 = b^2 + 8^2 - 2(8)b \cos 60^\circ$  gives  $b^2 - 8b + 15 = 0$ , so  $(b - 3)(b - 5) = 0$  and  $AC = 3$  or  $5$ . Both are valid triangles.

### Common mistakes

- Using the sine rule with an angle and a side that are *not* opposite each other.
- Applying  $\frac{1}{2}ab \sin C$  with an angle that is not between the two sides.
- Rejecting the obtuse solution in the ambiguous case without checking whether the angles still add to  $180^\circ$ .
- In the cosine rule, forgetting that  $-2bc \cos A$  becomes *positive* when  $\cos A$  is negative.

## 3 Radians, arcs, sectors and segments

Degrees are an arbitrary choice (360 comes from ancient astronomy). The natural unit of angle is defined by the circle itself. **One radian** is the angle at the centre of a circle subtended by an arc whose length equals the radius.



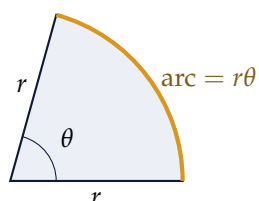
The whole circumference is  $2\pi r$ , which is  $2\pi$  radii, so a full turn is  $2\pi$  radians:

$$2\pi \text{ rad} = 360^\circ, \quad \pi \text{ rad} = 180^\circ, \quad 1 \text{ rad} = \frac{180^\circ}{\pi} \approx 57.3^\circ.$$

To convert, multiply degrees by  $\frac{\pi}{180}$ , or radians by  $\frac{180}{\pi}$ . Learn the common ones:  $30^\circ = \frac{\pi}{6}$ ,  $45^\circ = \frac{\pi}{4}$ ,  $60^\circ = \frac{\pi}{3}$ ,  $90^\circ = \frac{\pi}{2}$ ,  $180^\circ = \pi$ ,  $270^\circ = \frac{3\pi}{2}$ ,  $360^\circ = 2\pi$ . An angle given as a multiple of  $\pi$  (or with no unit at all) is in radians.

**Arc length.** An angle of  $\theta$  radians subtends an arc  $\theta$  times the radius, so  $s = r\theta$ . (Check:  $\theta = 2\pi$  gives  $2\pi r$ .) **Sector area.**

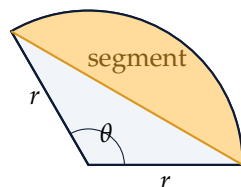
A sector is the fraction  $\frac{\theta}{2\pi}$  of a whole circle, so its area is  $\frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta$ . Both formulas are valid *only* when  $\theta$  is in radians; this is the reason radians are used.



$$\text{perimeter} = 2r + r\theta$$

$$\text{area} = \frac{1}{2}r^2\theta$$

**Segment.** A *segment* is the region between an arc and its chord. It is the sector minus the triangle formed by the two radii and the chord, and that triangle has area  $\frac{1}{2}r^2 \sin \theta$  (area formula, Section 2). So the segment area is  $\frac{1}{2}r^2(\theta - \sin \theta)$ .



$$\text{segment} = \text{sector} - \text{triangle}$$

$$= \frac{1}{2}r^2\theta - \frac{1}{2}r^2 \sin \theta$$

#### Worked example: converting

Convert  $150^\circ$  to radians and  $\frac{7\pi}{4}$  to degrees.

$$150 \times \frac{\pi}{180} = \frac{5\pi}{6}, \text{ and } \frac{7\pi}{4} \times \frac{180}{\pi} = 315^\circ.$$

#### Worked example: arc length and sector area

A sector has radius 12 cm and angle  $\frac{\pi}{3}$ . Find the arc length, the perimeter and the area.

$$\text{Arc} = 12 \times \frac{\pi}{3} = 4\pi \text{ cm. Perimeter} = 24 + 4\pi \text{ cm. Area} = \frac{1}{2}(12)^2 \frac{\pi}{3} = 24\pi \text{ cm}^2.$$

#### Worked example: a segment

A chord subtends an angle  $\frac{\pi}{3}$  at the centre of a circle of radius 6. Find the area of the minor segment.

$$\text{Sector} = \frac{1}{2}(36) \frac{\pi}{3} = 6\pi. \text{ Triangle} = \frac{1}{2}(36) \sin \frac{\pi}{3} = 9\sqrt{3}. \text{ Segment} = 6\pi - 9\sqrt{3} \approx 3.2. \text{ (A sense check: a small sliver, much less than the sector's 18.8.)}$$

### Common mistakes

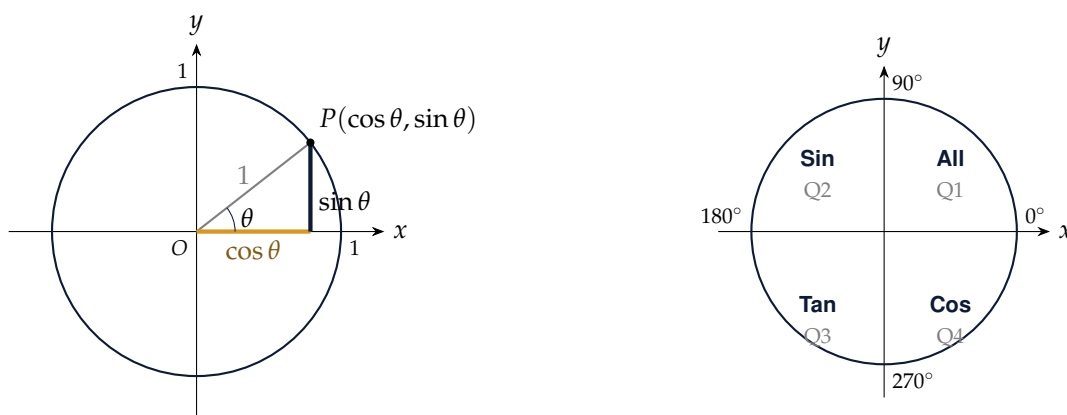
- Using  $s = r\theta$  or  $\frac{1}{2}r^2\theta$  with  $\theta$  in degrees. Convert first.
- Forgetting the two radii when asked for the *perimeter* of a sector.
- Forgetting that the triangle in a segment uses  $\frac{1}{2}r^2 \sin \theta$ , with  $\theta$  the angle at the centre.

## 4 Sine, cosine and tangent of any angle: the unit circle

In a right-angled triangle an angle can only be below  $90^\circ$ . To handle larger and negative angles we use a circle of radius 1 centred on the origin, the **unit circle**. Start at the point  $(1, 0)$  and turn anticlockwise through an angle  $\theta$  (clockwise if  $\theta$  is negative) to reach a point  $P$ . Then **define**

$$\cos \theta = x\text{-coordinate of } P, \quad \sin \theta = y\text{-coordinate of } P, \quad \tan \theta = \frac{\sin \theta}{\cos \theta}.$$

This agrees with the right-angled definition for acute angles: the radius is the hypotenuse 1, so adjacent =  $\cos \theta$  and opposite =  $\sin \theta$ .

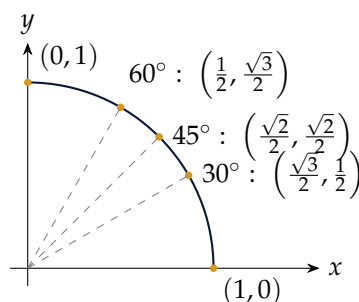


Everything now follows from the picture. Since  $P$  is on a circle of radius 1, both coordinates lie between  $-1$  and  $1$ , so

$$-1 \leq \sin \theta \leq 1, \quad -1 \leq \cos \theta \leq 1.$$

The sign of each coordinate depends on the quadrant. This is remembered as **CAST** (start in the bottom right and go anticlockwise): in the first quadrant **All** of sin, cos and tan are positive; in the second only **Sin**; in the third only **Tan**; in the fourth only **Cos**. The angles  $0^\circ, 90^\circ, 180^\circ, 270^\circ$  give the points  $(1, 0), (0, 1), (-1, 0), (0, -1)$ , so sin takes the values  $0, 1, 0, -1$  and cos takes  $1, 0, -1, 0$ . Because  $\tan \theta = \sin \theta / \cos \theta$ , it is undefined where  $\cos \theta = 0$ , that is at  $90^\circ$  and  $270^\circ$ .

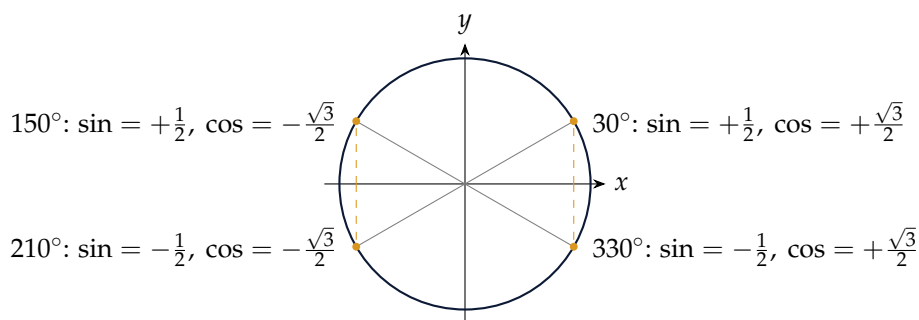
**Exact values.** The points at  $30^\circ, 45^\circ$  and  $60^\circ$  have exactly the coordinates from the special triangles:



**Reflection symmetry gives every other angle.** Reflect a point across the  $y$ -axis (angle  $\theta \rightarrow 180^\circ - \theta$ ): the height stays and the  $x$ -coordinate changes sign. Reflect across the  $x$ -axis (angle  $\theta \rightarrow -\theta$ ): the  $x$ -coordinate stays and the height changes sign. Rotating by  $180^\circ$  changes both signs. In symbols:

$$\begin{aligned} \sin(180^\circ - \theta) &= \sin \theta, & \cos(180^\circ - \theta) &= -\cos \theta, & \sin(180^\circ + \theta) &= -\sin \theta, & \cos(180^\circ + \theta) &= -\cos \theta, \\ \sin(-\theta) &= -\sin \theta, & \cos(-\theta) &= \cos \theta, & \sin(\theta + 360^\circ) &= \sin \theta, & \cos(\theta + 360^\circ) &= \cos \theta. \end{aligned}$$

In practice: find the acute **reference angle** between the point and the  $x$ -axis, take the value for that acute angle, and attach the sign from CAST.



### Worked example: exact values at any angle

Find  $\sin 225^\circ$ ,  $\cos \frac{5\pi}{6}$  and  $\tan 300^\circ$ .

$225^\circ$  is in the third quadrant with reference angle  $45^\circ$ ; sine is negative there, so  $\sin 225^\circ = -\frac{\sqrt{2}}{2}$ .  $\frac{5\pi}{6} = 150^\circ$  is in the second quadrant with reference angle  $30^\circ$ , and cosine is negative:  $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ .  $300^\circ$  is in the fourth quadrant with reference angle  $60^\circ$ ; tangent is negative there:  $\tan 300^\circ = -\sqrt{3}$ .

### Worked example: working back from a value

Find all angles  $\theta$  with  $0^\circ \leq \theta \leq 360^\circ$  for which  $\sin \theta = -\frac{1}{2}$ .

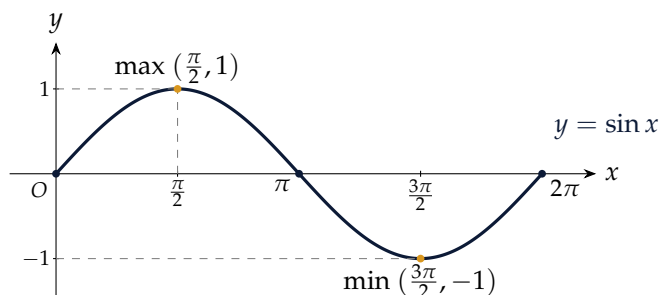
Reference angle:  $\sin 30^\circ = \frac{1}{2}$ . Sine is negative in the third and fourth quadrants, so  $\theta = 180^\circ + 30^\circ = 210^\circ$  and  $\theta = 360^\circ - 30^\circ = 330^\circ$ .

### Common mistakes

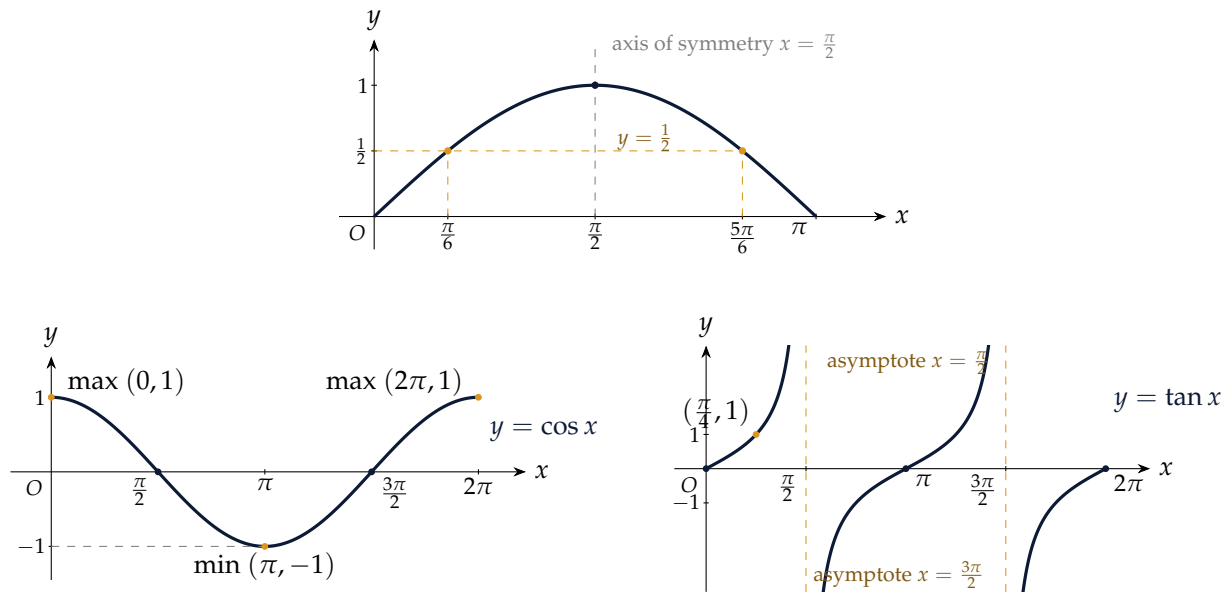
- Giving a positive value for sin or cos of an angle in a quadrant where it is negative. Always sketch the quadrant.
- Using the reference angle itself as the answer:  $\sin 210^\circ$  is  $-\frac{1}{2}$ , not  $+\frac{1}{2}$  and the angle  $210^\circ$  is not  $30^\circ$ .
- Thinking  $\tan 90^\circ = \infty$  is a number. It is undefined.

## 5 The graphs of sine, cosine and tangent, and their transformations

Plot  $\sin \theta$  (the height of  $P$ ) against the angle  $\theta$  as  $P$  goes round the circle. Starting at 0, the height rises to 1 at  $\frac{\pi}{2}$ , falls through 0 at  $\pi$  to  $-1$  at  $\frac{3\pi}{2}$ , and returns to 0 at  $2\pi$ , after which everything repeats.



Key facts for  $y = \sin x$ : it is **periodic** with period  $2\pi$  (it repeats every  $2\pi$ ); it oscillates between  $-1$  and  $1$  (**amplitude 1**); it crosses the axis at multiples of  $\pi$ ; and it is symmetric about every maximum, in particular about  $x = \frac{\pi}{2}$ , which is the statement  $\sin(\pi - x) = \sin x$ . In the figure, the two points where  $y = \frac{1}{2}$  are at  $\frac{\pi}{6}$  and  $\frac{5\pi}{6}$ , equal distances either side of  $\frac{\pi}{2}$ .



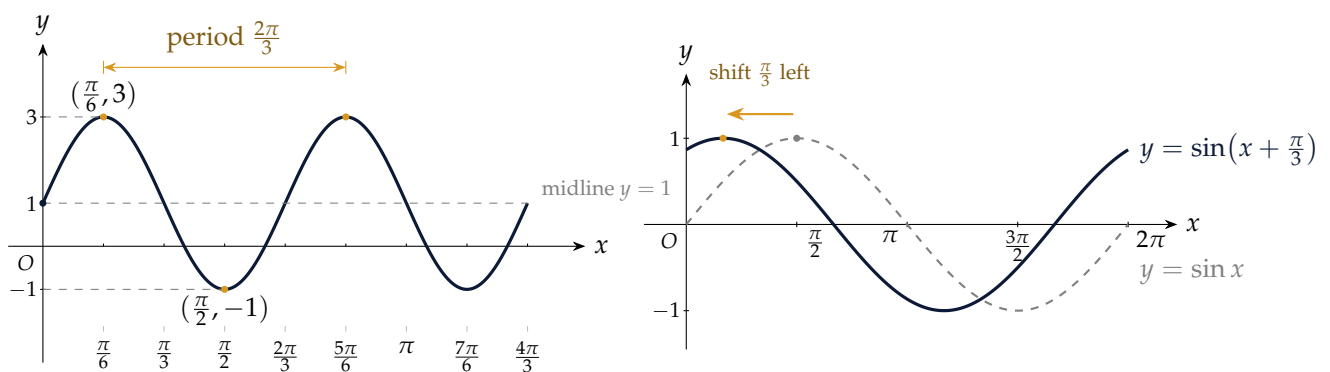
The graph of  $y = \cos x$  is the *same shape* as sine shifted  $\frac{\pi}{2}$  to the left: it starts at its maximum 1 at  $x = 0$  (because  $P$  starts at  $(1, 0)$ ), crosses the axis at  $\frac{\pi}{2}$ ,  $\frac{3\pi}{2}$ , and has period  $2\pi$ . It is symmetric about the  $y$ -axis ( $\cos(-x) = \cos x$ ), while sine has half-turn symmetry about the origin ( $\sin(-x) = -\sin x$ ).

The graph of  $y = \tan x$  is quite different: it has **asymptotes** (vertical lines it approaches but never reaches) at  $x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$  where  $\cos x = 0$ , it passes through the origin and through  $(\frac{\pi}{4}, 1)$ , it takes *every* value, and its period is only  $\pi$ .

**Transformations.** The general sine wave  $y = a \sin(bx + c) + d$  has

- amplitude  $|a|$  (a stretch of the height by the factor  $a$ ; a negative  $a$  also flips the curve);
- period  $\frac{2\pi}{b}$  (the factor  $b$  squeezes the curve horizontally, so a larger  $b$  makes more waves);
- a horizontal shift of  $\frac{c}{b}$  to the *left* (for  $y = \sin(x + c)$  the curve moves left by  $c$ : the value at  $x$  equals the old value at  $x + c$ );
- a vertical shift of  $d$ , so the midline is  $y = d$ , the maximum is  $d + |a|$  and the minimum  $d - |a|$ .

The same rules hold for cosine, and for tangent (with period  $\pi/b$ ).



#### Worked example: reading a transformed graph

State the amplitude, period, midline, maximum and minimum of  $y = 2 \sin 3x + 1$  (shown above).

Amplitude 2; period  $\frac{2\pi}{3}$ ; midline  $y = 1$ ; maximum  $1 + 2 = 3$  (first reached at  $3x = \frac{\pi}{2}$ , i.e.  $x = \frac{\pi}{6}$ ); minimum  $1 - 2 = -1$ .

### Worked example: a horizontal shift

Where is the first maximum of  $y = \sin\left(x + \frac{\pi}{3}\right)$  for  $x > 0$ ?

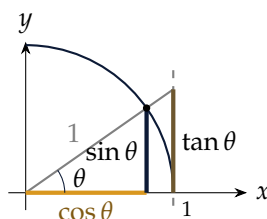
The maximum of sine occurs when the bracket is  $\frac{\pi}{2}$ :  $x + \frac{\pi}{3} = \frac{\pi}{2}$ , so  $x = \frac{\pi}{6}$ . This is  $\frac{\pi}{3}$  to the left of the normal first maximum at  $\frac{\pi}{2}$ .

### Common mistakes

- Reading the period of  $\sin bx$  as  $b \times 2\pi$  or  $\frac{b}{2\pi}$ . It is  $\frac{2\pi}{b}$ .
- Shifting the wrong way:  $\sin(x + c)$  moves *left*.
- Giving the amplitude of  $y = 2\sin 3x + 1$  as 3 or 4: it is the height from the midline, 2.
- Drawing tangent with a maximum or with range  $[-1, 1]$ .

## 6 The two basic identities

An *identity* is an equation true for every angle. Two follow directly from the unit circle.



**Identity 1.** The point  $P(\cos \theta, \sin \theta)$  is on the circle  $x^2 + y^2 = 1$ , so

$$\sin^2 \theta + \cos^2 \theta = 1.$$

(In a right-angled triangle this is Pythagoras' theorem divided by the hypotenuse squared.) It lets you swap  $\sin^2 \theta$  for  $1 - \cos^2 \theta$ , or  $\cos^2 \theta$  for  $1 - \sin^2 \theta$ .

**Identity 2.** By definition  $\tan \theta = \frac{\sin \theta}{\cos \theta}$ . In the figure, the similar triangles give the vertical line at  $x = 1$  a height of  $\frac{\sin \theta}{\cos \theta}$ : that line segment *is* the tangent.

Writing  $\sin^2 \theta$  means  $(\sin \theta)^2$ . The usual strategies: (i) if only one of sine and cosine is known, use Identity 1 to get the other, choosing the sign from the quadrant; (ii) to simplify, write everything in terms of sin and cos; (iii) to solve an equation containing both  $\sin^2 x$  and  $\cos x$  (or vice versa), use Identity 1 to leave only one function.

### Worked example: finding the other ratios

$\sin \theta = \frac{3}{5}$  and  $\theta$  is obtuse. Find  $\cos \theta$  and  $\tan \theta$ .

$\cos^2 \theta = 1 - \frac{9}{25} = \frac{16}{25}$ , so  $\cos \theta = \pm \frac{4}{5}$ . An obtuse angle is in the second quadrant, where cosine is negative, so

$$\cos \theta = -\frac{4}{5} \text{ and } \tan \theta = \frac{3/5}{-4/5} = -\frac{3}{4}.$$

### Worked example: simplifying

Simplify  $\frac{1 - \cos^2 x}{\sin x}$  and  $\frac{\sin x}{\tan x}$  (where defined).

$1 - \cos^2 x = \sin^2 x$ , so the first is  $\frac{\sin^2 x}{\sin x} = \sin x$ . For the second,  $\frac{\sin x}{\sin x / \cos x} = \cos x$ .

### Worked example: using a sum

If  $\sin \theta + \cos \theta = \frac{6}{5}$ , find  $\sin \theta \cos \theta$ .

Square both sides:  $\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta = \frac{36}{25}$ . The outer terms sum to 1 by Identity 1, so  $2 \sin \theta \cos \theta = \frac{11}{25}$  and  $\sin \theta \cos \theta = \frac{11}{50}$ . Squaring is the key move: it creates the product  $\sin \theta \cos \theta$  and Identity 1 turns the rest into a number.

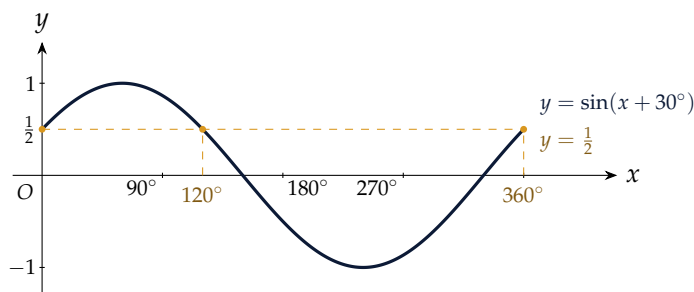
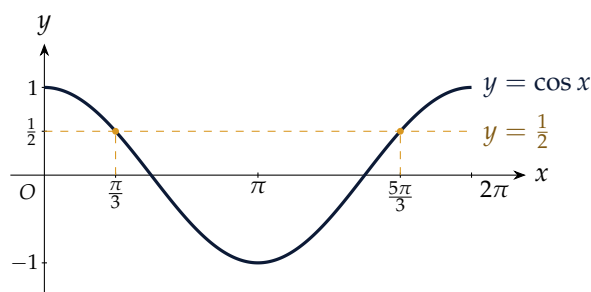
### Common mistakes

- Writing  $\sin^2 \theta + \cos^2 \theta = 1$  as  $\sin \theta + \cos \theta = 1$ , or as  $\sin^2 \theta + \cos^2 \theta$  equal to something else.
- Taking the positive square root when finding  $\cos \theta$  from  $\sin \theta$ : check the quadrant.
- Cancelling a common factor containing  $\sin x$  or  $\cos x$  in an equation (see Section 7): it may be zero.

## 7 Solving trigonometric equations in an interval

A trigonometric equation such as  $\cos x = \frac{1}{2}$  has infinitely many solutions because the functions repeat, so a question gives an *interval* (such as  $0 \leq x \leq 2\pi$ ) and asks for every solution inside it. The reliable method:

1. Rearrange to isolate  $\sin(\dots)$ ,  $\cos(\dots)$  or  $\tan(\dots)$ , and if there is a quadratic or a mix of functions, use Section 6 to reduce it to one function and factorise.
2. Find the *reference angle* from the exact values (ignore the sign of the right-hand side), then use CAST or the graph to list every angle in one period with the required sign.
3. If the angle inside is  $bx + c$  rather than  $x$ , first convert the interval for  $x$  into the interval for  $bx + c$ , list all solutions there, then solve back for  $x$ .
4. Check every solution against the original equation and the interval.



The graph shows why there are two solutions in  $[0, 2\pi]$ : the line  $y = \frac{1}{2}$  meets the cosine curve at  $x = \frac{\pi}{3}$  and at its mirror image  $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$ .

### Worked example: a basic equation

Solve  $2 \cos x = 1$  for  $0 \leq x \leq 2\pi$  (shown above), and  $\tan x = -1$  for  $0^\circ \leq x \leq 360^\circ$ .

$\cos x = \frac{1}{2}$ : reference angle  $\frac{\pi}{3}$ , cosine positive in the first and fourth quadrants, so  $x = \frac{\pi}{3}, \frac{5\pi}{3}$ . For  $\tan x = -1$  the reference angle is  $45^\circ$  and tangent is negative in the second and fourth quadrants:  $x = 135^\circ, 315^\circ$  (tangent repeats every  $180^\circ$ ).

### Worked example: a shifted angle

Solve  $\sin(x + 30^\circ) = \frac{1}{2}$  for  $0^\circ \leq x \leq 360^\circ$  (shown above).

Let  $u = x + 30^\circ$ . Then  $30^\circ \leq u \leq 390^\circ$ . In that range  $\sin u = \frac{1}{2}$  at  $u = 30^\circ, 150^\circ, 390^\circ$ . Subtracting  $30^\circ$ :  $x = 0^\circ, 120^\circ, 360^\circ$ . Note the *extra* solution at the end, found only because the interval for  $u$  was extended.

### Worked example: a multiple angle

Solve  $\sin 2x = \frac{\sqrt{3}}{2}$  for  $0^\circ \leq x \leq 360^\circ$ .

Let  $u = 2x$ , so  $0^\circ \leq u \leq 720^\circ$ : two full turns.  $\sin u = \frac{\sqrt{3}}{2}$  at  $u = 60^\circ, 120^\circ, 420^\circ, 480^\circ$ , so  $x = 30^\circ, 60^\circ, 210^\circ, 240^\circ$ : four solutions. In general  $\sin bx$  has  $b$  times as many solutions.

#### Worked example: a quadratic in $\sin x$

Solve  $2\sin^2 x - \sin x - 1 = 0$  for  $0^\circ \leq x \leq 360^\circ$ .

Treat  $s = \sin x$  as the unknown:  $2s^2 - s - 1 = (2s + 1)(s - 1) = 0$ , so  $\sin x = -\frac{1}{2}$  or  $\sin x = 1$ . The first gives  $x = 210^\circ, 330^\circ$  and the second  $x = 90^\circ$ . Solutions:  $90^\circ, 210^\circ, 330^\circ$ .

#### Worked example: never divide by a function

Solve  $\sin x = 2 \sin x \cos x$  for  $0^\circ \leq x \leq 360^\circ$ .

Dividing by  $\sin x$  would lose every solution where  $\sin x = 0$ . Instead move everything to one side and factorise:  $\sin x(1 - 2 \cos x) = 0$ . Either  $\sin x = 0$ , giving  $x = 0^\circ, 180^\circ, 360^\circ$ , or  $\cos x = \frac{1}{2}$ , giving  $x = 60^\circ, 300^\circ$ . Five solutions in all.

#### Common mistakes

- Finding only the first solution: always look for the others in the interval (the mirror image, and one period later).
- Forgetting to change the interval when solving  $\sin(bx + c) = k$ : you lose solutions at the ends.
- Whether an end-point is included.  $0 \leq x \leq 2\pi$  counts both 0 and  $2\pi$ ;  $0 \leq x < 2\pi$  counts only the first.
- Using degrees on a calculator-free answer where the question is in radians, or vice versa.

## 8 Strategy, bounds and applied problems

**Choosing a method.** Right angle present: SOH-CAH-TOA. Otherwise, a matching side-angle pair: sine rule; two sides and the included angle, or three sides: cosine rule; area from two sides and the included angle:  $\frac{1}{2}ab \sin C$ . For regular shapes and anything circular, split into triangles at the centre. Always sketch, and check that the largest angle faces the largest side.

**Intuitive idea 1: use the bounds.** Since  $-1 \leq \sin x \leq 1$  and  $-1 \leq \cos x \leq 1$ , the greatest and least values of an expression come from the extremes of the sine or cosine inside it, without any equation-solving. For  $y = \frac{1}{2 - \cos x}$ , the denominator lies between 1 and 3, so  $y$  is greatest ( $= 1$ ) when  $\cos x = 1$  and least ( $= \frac{1}{3}$ ) when  $\cos x = -1$ . Questions about a periodic quantity (tide height, temperature, a point on a wheel) are the same thing: it oscillates between  $d \pm a$ , and the time spent above a level is found from the equation  $\sin(\cdot) = k$  and its mirror solution.

**Intuitive idea 2: regular polygons.** A regular  $n$ -sided polygon inscribed in a circle of radius  $r$  splits into  $n$  identical isosceles triangles at the centre, each with apex angle  $\frac{2\pi}{n}$  and area  $\frac{1}{2}r^2 \sin \frac{2\pi}{n}$ .

#### Worked example: a periodic quantity

The temperature in a greenhouse is  $T = 15 + 5 \sin \frac{\pi t}{12}$  degrees Celsius, where  $t$  is the time in hours after midnight,  $0 \leq t \leq 24$ . For how long is it above  $17.5^\circ\text{C}$ ?

$T > 17.5$  when  $5 \sin \frac{\pi t}{12} > 2.5$ , i.e.  $\sin u > \frac{1}{2}$  with  $u = \frac{\pi t}{12}$  in  $[0, 2\pi]$ . That happens for  $\frac{\pi}{6} < u < \frac{5\pi}{6}$ , so  $2 < t < 10$ : eight hours. (The maximum,  $20^\circ\text{C}$ , is at  $t = 6$  and the coldest,  $10^\circ\text{C}$ , at  $t = 18$ .)

#### Worked example: two angles of elevation

From a point  $A$  on level ground the angle of elevation of the top of a tower is  $45^\circ$ . From a point  $B$ , 10 m further from the tower in the same line, it is  $30^\circ$ . Find the height of the tower.

Let the height be  $h$  and the distance from  $A$  to the foot be  $x$ . Then  $\tan 45^\circ = \frac{h}{x}$  gives  $x = h$ , and  $\tan 30^\circ = \frac{h}{x + 10}$  gives  $x + 10 = \sqrt{3}h$ . So  $h + 10 = \sqrt{3}h$  and  $h = \frac{10}{\sqrt{3} - 1} = 5(\sqrt{3} + 1) \approx 13.7$  m.

### Worked example: two ships

Two ships leave port together, one sailing 10 km on a bearing of  $100^\circ$  and the other 6 km on a bearing of  $040^\circ$ . How far apart are they?

The angle between the paths is  $100^\circ - 40^\circ = 60^\circ$ . Cosine rule:  $d^2 = 10^2 + 6^2 - 2(10)(6) \cos 60^\circ = 136 - 60 = 76$ , so  $d = 2\sqrt{19} \approx 8.7$  km.

### Exam tip

A negative cosine in the cosine rule signals an obtuse angle, and a sine equal to a number greater than 1 signals that no such triangle exists. When a question asks “how many” solutions, sketch the curve and the horizontal line rather than solving.

## Check yourself

Try these without looking back, then compare with the answers on the next page.

- Find  $\cos \theta$  and  $\tan \theta$  if  $\sin \theta = \frac{5}{13}$  and  $\theta$  is acute.
- Write down the exact values of  $\sin 150^\circ$ ,  $\cos 240^\circ$  and  $\tan 135^\circ$ .
- In triangle  $ABC$ ,  $A = 60^\circ$ ,  $a = 6$  and  $B = 45^\circ$ . Find  $b$ .
- A triangle has sides 4, 5 and 6. Find  $\cos$  of its largest angle.
- Find the area of a triangle with sides 6 and 10 and an included angle of  $30^\circ$ .
- A sector has radius 5 and angle  $\frac{2\pi}{5}$ . Find its arc length and area.
- State the amplitude, period and range of  $y = 4 \cos 2x - 1$ .
- Solve  $2 \sin x = -\sqrt{2}$  for  $0^\circ \leq x < 360^\circ$ .

## Answers to the self-check

1.  $\cos \theta = \frac{12}{13}$  (from  $1 - \frac{25}{169} = \frac{144}{169}$ ) and  $\tan \theta = \frac{5}{12}$ .
2.  $\frac{1}{2}$ ,  $-\frac{1}{2}$  and  $-1$ .
3.  $b = \frac{6 \sin 45^\circ}{\sin 60^\circ} = \frac{6(\sqrt{2}/2)}{\sqrt{3}/2} = 2\sqrt{6}$ .
4.  $\frac{16 + 25 - 36}{2 \cdot 4 \cdot 5} = \frac{1}{8}$ .
5.  $\frac{1}{2}(6)(10) \sin 30^\circ = 15$ .
6. Arc  $= 5 \cdot \frac{2\pi}{5} = 2\pi$ ; area  $= \frac{1}{2}(25)\frac{2\pi}{5} = 5\pi$ .
7. Amplitude 4, period  $\pi$ , range  $-5 \leq y \leq 3$ .
8.  $\sin x = -\frac{\sqrt{2}}{2}$ :  $x = 225^\circ, 315^\circ$ .

### Next step

Heurisco's *ESAT Maths 2: Trigonometry Practice Set* has twenty original exam-style questions on this lesson, with diagrams, in order of difficulty, with a full worked solution for each.