

ESAT

Maths 2

Exponentials and logarithms

Free lesson notes

Covers

MM5.1 to MM5.3

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

This lesson covers the exponential function $y = a^x$ and its graph, the meaning of a logarithm, the laws of logarithms, and solving equations that involve indices or logarithms, including equations that reduce to a quadratic. The central idea is that $\log_a$ is the *inverse* of a^x : a logarithm is an index.

How to use it. Try each worked example first. Then do the self-check, then the practice set. In the test there is no calculator, so answers are always exact: either whole numbers, fractions, or logarithms left in exact form.

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1 Exponential graphs and the meaning of a logarithm

The graph of $y = a^x$ (with $a > 1$) passes through $(0, 1)$, is always positive, increases ever more steeply, and has the x -axis ($y = 0$) as a horizontal asymptote. For $0 < a < 1$ it is a decaying curve with the same asymptote. A logarithm answers the question “what power?”:

$$a^b = c \iff b = \log_a c.$$

So $\log_2 8 = 3$ because $2^3 = 8$. Also $\log_a 1 = 0$ and $\log_a a = 1$. Transformations shift the asymptote: $y = 2^{x+1} - 5$ has asymptote $y = -5$ and meets the y -axis at $2^1 - 5 = -3$.

Worked example: converting forms

Write $3^x = 7$ in logarithmic form, and find $\log_{25} 125$.

$x = \log_3 7$. For the second, let $\log_{25} 125 = b$, so $25^b = 125$, i.e. $5^{2b} = 5^3$, giving $b = \frac{3}{2}$.

Worked example: a shifted exponential

Find the y -intercept and the asymptote of $y = 2^{x+1} - 5$.

At $x = 0$: $y = 2 - 5 = -3$. As $x \rightarrow -\infty$, $2^{x+1} \rightarrow 0$, so the asymptote is $y = -5$.

Common mistakes

- Writing $\log_2 8 = \frac{8}{2}$: a logarithm is an index, not a quotient.
- Thinking an exponential graph can cross the x -axis: $a^x > 0$ for every x , so $y = a^x$ has no real roots.
- Forgetting that the asymptote of $y = a^x + c$ is $y = c$, not $y = 0$.

2 The laws of logarithms

For $m, n > 0$:

$$\log_a m + \log_a n = \log_a(mn), \quad \log_a m - \log_a n = \log_a\left(\frac{m}{n}\right), \quad \log_a(m^k) = k \log_a m.$$

Changing base: $\log_a b = \frac{\log_c b}{\log_c a}$, which gives $\log_a b \times \log_b a = 1$. Each law is the corresponding law of indices in disguise. There is *no* rule for $\log(m + n)$.

Worked example: combining logs

Write $2\log_a 3 + \log_a 4 - \log_a 6$ as a single logarithm.

$$\log_a 9 + \log_a 4 - \log_a 6 = \log_a \frac{9 \times 4}{6} = \log_a 6.$$

Worked example: change of base

Evaluate $\log_2 3 \times \log_3 8$.

$$\log_2 3 \times \log_3 8 = \frac{\log_2 3}{1} \times \frac{\log_2 8}{\log_2 3} = \log_2 8 = 3.$$

Common mistakes

- Writing $\log(m + n) = \log m + \log n$. The law is for the *product*: $\log m + \log n = \log(mn)$.
- Writing $\frac{\log m}{\log n} = \log(m - n)$. A quotient of logarithms is a change of base, not a difference.
- Using the laws when the arguments are not positive.

3 Solving exponential and logarithmic equations

- **Same base:** write both sides as powers of one base and equate the indices, e.g. $3^{2x-1} = 27 = 3^3$.
- **Different bases:** take logarithms, or express as $x = \log_a b$.
- **Log equations:** combine into one logarithm, convert to an index equation, solve, and then *check every answer* in the original equation: the argument of a logarithm must be positive.

Worked example: a log equation

Solve $\log_2 x + \log_2(x - 2) = 3$.

$\log_2(x(x - 2)) = 3$, so $x^2 - 2x = 8$ and $(x - 4)(x + 2) = 0$. In the original equation $x > 2$, so $x = -2$ is rejected. The solution is $x = 4$.

Worked example: a reducible equation

Solve $4^x - 5 \cdot 2^x + 4 = 0$.

Let $u = 2^x$, so $u^2 - 5u + 4 = 0$ and $(u - 1)(u - 4) = 0$. Then $2^x = 1$ gives $x = 0$, and $2^x = 4$ gives $x = 2$.

Common mistakes

- Not checking that the arguments of the logarithms are positive. $x = -2$ above makes $\log_2 x$ undefined.
- Writing $\log_2 x + \log_2(x - 2) = 3$ as $\log_2 x \cdot \log_2(x - 2) = 3$.
- Rejecting $2^x = -1$ is correct, but rejecting a perfectly good positive value of u loses a solution.

4 Growth, decay and modelling

Many situations follow $P = P_0 b^{kt}$: growth if $b > 1$, decay if $0 < b < 1$. To find a time, set up the equation and write both sides in the same base. The numbers in test questions are chosen so that the index comes out cleanly (e.g. $2^{t/3} = 8$ gives $t = 9$).

Worked example: doubling time

A population is modelled by $P = 500 \times 2^{t/3}$, where t is in years. When does it reach 4000?

$$2^{t/3} = 8 = 2^3, \text{ so } \frac{t}{3} = 3 \text{ and } t = 9 \text{ years.}$$

Worked example: two linked indices

Given $2^x = 3$ and $3^y = 4$, find xy .

$$4 = 3^y = (2^x)^y = 2^{xy}, \text{ so } xy = 2 \text{ because } 4 = 2^2.$$

Common mistakes

- Mixing up the rate and the starting value: $P = P_0 b^{kt}$ has P_0 at $t = 0$.
- Dividing the exponent instead of equating it: if $2^{t/3} = 2^3$ then $t/3 = 3$.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Write $\log_3 27$ and $\log_5 \frac{1}{25}$ as integers.
2. Simplify $\log_2 10 + \log_2 3 - \log_2 15$.
3. Solve $2^{3x+1} = 32$.
4. Solve $\log_3(x + 2) = 2$.
5. Solve $9^x - 4 \cdot 3^x + 3 = 0$.
6. State the asymptote and y -intercept of $y = 3^x - 2$.

Answers to the self-check

1. 3 and -2 .
2. $\log_2 \frac{10 \times 3}{15} = \log_2 2 = 1$.
3. $3x + 1 = 5$, so $x = \frac{4}{3}$.
4. $x + 2 = 9$, so $x = 7$.
5. $u = 3^x$: $(u - 1)(u - 3) = 0$, so $x = 0$ or $x = 1$.
6. Asymptote $y = -2$; y -intercept $1 - 2 = -1$.

Next step

Heurisco's *ESAT Maths 2: Exponentials and Logarithms Practice Set* has twenty original exam-style questions on this lesson, in order of difficulty, with a full worked solution for each.