

ESAT

Maths 2

Differentiation

Free lesson notes

Covers

MM6.1 to MM6.3

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

Differentiation answers one question: *how steep is a curve at a particular point?* The same idea tells us how fast anything is changing: the speed of a car, the rate at which a balloon's volume grows as it is inflated, the way a profit rises and then falls. Once you can find a gradient you can also find the highest and lowest points of a curve, which is the key to every "greatest" and "least" problem.

This lesson builds the subject from scratch. We begin with the problem of defining the gradient of a curve, and solve it by sliding a chord towards a tangent. From that picture we derive the rule for differentiating powers of x rather than just quoting it, then use it (with the rewriting steps that cause most of the mistakes). We then look at rates of change and the second derivative, tangents and normals, stationary points and what they tell us about the shape of a curve, and finally optimisation. The last section collects intuitive checks that ESAT likes to test.

How to use it. Read each idea, then cover the worked example and try it yourself. Always sketch the curve when you can: a picture often shows whether an answer is sensible. Finish with the self-check and then the practice set. Specification codes: MM6.1 (the derivative as a gradient and as a rate of change, the second derivative, notation), MM6.2 (differentiating x^n and sums) and MM6.3 (tangents, normals, stationary points, increasing and decreasing).

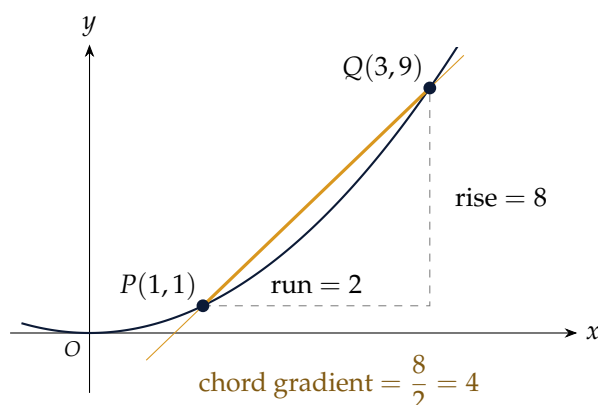
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1 The gradient of a curve

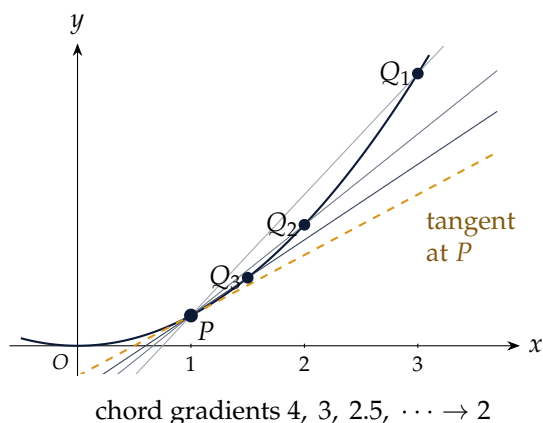
For a straight line the gradient is the same everywhere: rise divided by run. A curve is different. It bends, so it is steeper in some places than others, and we cannot talk about "the" gradient of a curve, only its gradient *at a point*. What should that mean?

Zoom in on a smooth curve at a point P and it looks more and more like a straight line. We define the gradient of the curve at P to be the gradient of that line, called the **tangent** to the curve at P . The difficulty is that a gradient needs two points, and we only have one.

The way out is a trick. Pick a second point Q on the curve and join P to Q . The straight line PQ is called a **chord**, and we can find its gradient from the coordinates of the two points. For the curve $y = x^2$ with $P = (1, 1)$ and $Q = (3, 9)$ the rise is 8 and the run is 2, so the chord has gradient 4.



That is not the gradient of the curve at P : the chord is too steep, because the curve is still climbing steeply between P and Q . But watch what happens when we slide Q along the curve towards P .



As Q moves closer, the chord swings round and settles onto the tangent at P . The gradients of the chords are 4, 3, 2.5, and if we keep going we get a pattern:

x -coordinate of Q	3	2	1.5	1.1	1.01
gradient of PQ	4	3	2.5	2.1	2.01

The chord gradients are closing in on 2, so the gradient of $y = x^2$ at $x = 1$ is 2. We never put Q exactly on P (that would be $\frac{0}{0}$), but we can see the number the gradients are heading towards.

Worked example: chords of $y = x^2$ from the point $(2, 4)$

Find the gradient of the chord from $P(2, 4)$ to Q on $y = x^2$ with x -coordinate $2 + h$, and say what happens as h gets very small.

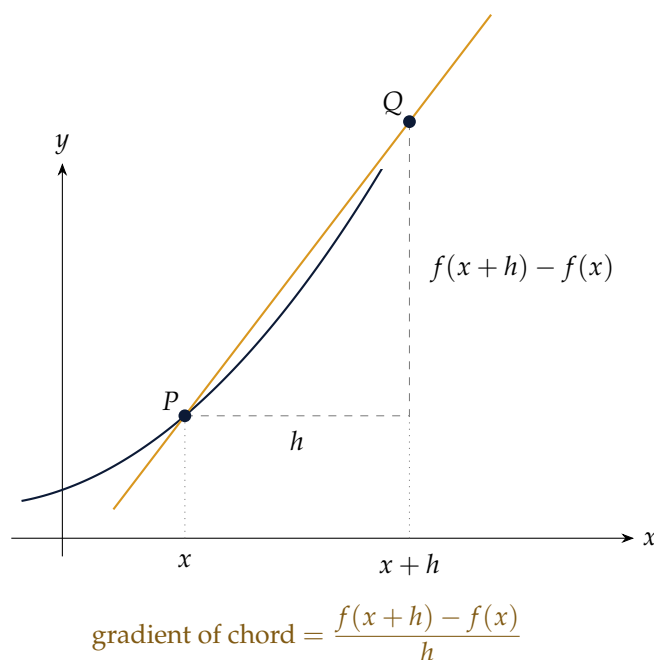
$Q = (2 + h, (2 + h)^2)$. The rise is $(2 + h)^2 - 4 = 4h + h^2$ and the run is h , so the gradient is $\frac{4h + h^2}{h} = 4 + h$. For $h = 1$ this is 5, for $h = 0.1$ it is 4.1, for $h = 0.001$ it is 4.001. As h shrinks the gradient approaches 4, so the curve has gradient 4 at $x = 2$.

Common mistakes

- Thinking a chord's gradient is the gradient of the curve. It is only an *approximation*, and it gets better as Q approaches P .
- Trying to put Q exactly on P straight away. The run would be 0 and you cannot divide by 0: first simplify, *then* let h become small.

2 First principles: deriving the rule

Let us do the chord argument once and for all with general letters. Let $P = (x, f(x))$ be any point on the curve $y = f(x)$, and let Q be a little further along, at horizontal distance h from P , so $Q = (x + h, f(x + h))$.



The run from P to Q is h , and the rise is $f(x+h) - f(x)$, so

$$\text{gradient of chord } PQ = \frac{f(x+h) - f(x)}{h}.$$

The **derivative** of f , written $f'(x)$ or $\frac{dy}{dx}$ (read “dee y by dee x ”), is the number this approaches as h becomes smaller and smaller. It is the gradient of the curve at x , and it is a *function*: put in a value of x , get out the gradient there.

Case $f(x) = x^2$. The chord gradient is $\frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h$. As $h \rightarrow 0$ this tends to $2x$, so $\frac{d}{dx}(x^2) = 2x$. At $x = 1$ it gives 2 and at $x = 2$ it gives 4, matching what we found by sliding chords.

Case $f(x) = x^3$. Expanding $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$, the chord gradient is $\frac{3x^2h + 3xh^2 + h^3}{h} = 3x^2 + 3xh + h^2 \rightarrow 3x^2$.

The pattern is $x^1 \rightarrow 1$, $x^2 \rightarrow 2x$, $x^3 \rightarrow 3x^2$. In general, when $(x+h)^n$ is expanded the first two terms are $x^n + nx^{n-1}h$ and every remaining term contains at least h^2 . The x^n cancels with the $-x^n$ in the numerator, and dividing by h leaves nx^{n-1} plus terms that still contain h and vanish. That proves the rule for whole-number powers.

Key ideas

The power rule. For any rational number n ,

$$\frac{d}{dx}(x^n) = nx^{n-1} : \text{ bring the power down as a multiplier, then reduce the power by one.}$$

Also: a constant has derivative 0 (its graph is flat); $\frac{d}{dx}(ax) = a$; and differentiation works *term by term*, with constants carried through:

$$\frac{d}{dx}(af(x) + bg(x)) = af'(x) + bg'(x).$$

The last property holds because the chord gradient of $af + bg$ is exactly a times the chord gradient of f plus b times that of g .

Does the rule really work for negative and fractional powers? Test it from first principles. For $f(x) = \frac{1}{x} = x^{-1}$:

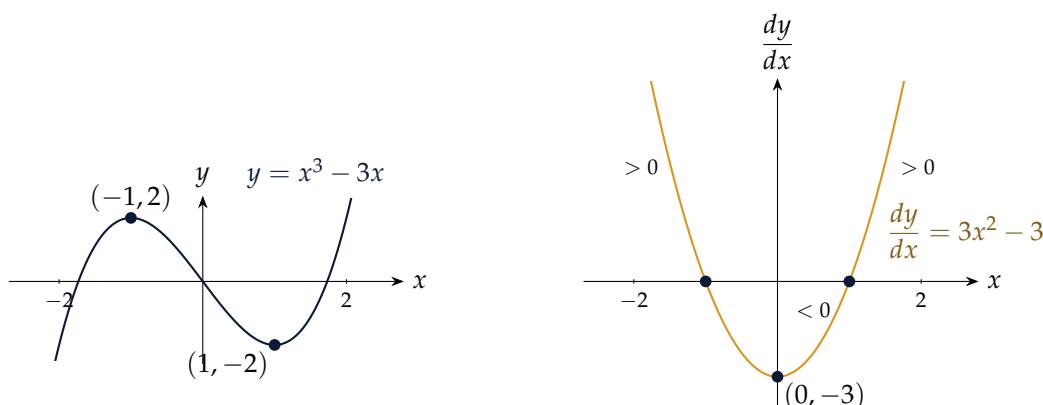
$$\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \frac{x - (x+h)}{hx(x+h)} = \frac{-1}{x(x+h)} \rightarrow -\frac{1}{x^2} = -x^{-2}, \text{ exactly what } nx^{n-1} \text{ gives with } n = -1. \text{ For } f(x) = \sqrt{x}, \text{ multiply top and bottom by } \sqrt{x+h} + \sqrt{x}: \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{1}{\sqrt{x+h} + \sqrt{x}} \rightarrow \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}, \text{ again the rule with } n = \frac{1}{2}.$$

Worked example: the derivative as a function

Use the rule to find the gradient of $y = x^3 - 3x$ at $x = 0$, $x = 1$ and $x = 2$, and say where the curve is flat.

$\frac{dy}{dx} = 3x^2 - 3$. At $x = 0$ the gradient is -3 (sloping down), at $x = 1$ it is 0 (flat), at $x = 2$ it is 9 (steep and rising).

The curve is flat where $3x^2 - 3 = 0$, that is at $x = \pm 1$.



The lower graph is the *gradient function* $\frac{dy}{dx} = 3x^2 - 3$. It is positive where the upper curve is rising, negative where it is falling, and zero at the turning points $(-1, 2)$ and $(1, -2)$. Reading one graph from the other is a skill worth practising.

Common mistakes

- Using the wrong sign in $\frac{(x+h)^2 - x^2}{h}$: the x^2 terms cancel, leaving $2xh + h^2$ in the numerator.
- Differentiating the constant term: $\frac{d}{dx}(5) = 0$, not 5.
- Writing $\frac{dy}{dx}$ as a fraction that can be cancelled. It is one symbol, read as a single quantity.

3 Differentiating in practice: rewrite first

The power rule only applies to terms of the form $(\text{constant}) \times x^n$. Almost every mistake in differentiation comes from skipping the *rewriting step* that gets an expression into that form.

- Roots and reciprocals become powers: $\sqrt{x} = x^{1/2}$, $\sqrt[3]{x} = x^{1/3}$, $\frac{1}{x} = x^{-1}$, $\frac{1}{x^2} = x^{-2}$, $\frac{1}{\sqrt{x}} = x^{-1/2}$, and $\frac{a}{x^n} = ax^{-n}$ (the constant stays on top).
- Brackets are *expanded*. There is no rule that lets you differentiate the factors of a product separately.
- A fraction with a single term underneath is split: $\frac{x^3 + 4x}{x} = x^2 + 4$ (divide each term by x).

After differentiating, a negative or fractional power can be turned back into a reciprocal or root when you substitute a value.

Worked example: a reciprocal root

Find the gradient of $y = \frac{4}{\sqrt{x}}$ at $x = 4$.

$y = 4x^{-1/2}$, so $\frac{dy}{dx} = 4 \times \left(-\frac{1}{2}\right) x^{-3/2} = -2x^{-3/2}$. At $x = 4$: $x^{3/2} = 4^{3/2} = 8$, so the gradient is $-\frac{2}{8} = -\frac{1}{4}$.

Worked example: expand, then differentiate

Differentiate $y = (2x + 1)(x - 3)$ and find the gradient at $x = 3$.

$y = 2x^2 - 6x + x - 3 = 2x^2 - 5x - 3$, so $\frac{dy}{dx} = 4x - 5$. At $x = 3$ the gradient is 7. (Multiplying the factors' derivatives, $2 \times 1 = 2$, would be wrong.)

Worked example: a fraction

Differentiate $y = \frac{x^3 + 4x}{x}$ for $x \neq 0$.

Split it: $y = x^2 + 4$, so $\frac{dy}{dx} = 2x$. (It would be wrong to differentiate the top and bottom separately and divide: that gives $3x^2 + 4$.)

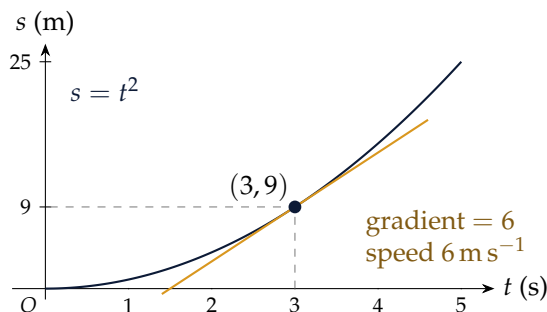
Common mistakes

- Differentiating $\frac{4}{\sqrt{x}}$ as $\frac{4}{\frac{1}{2}x^{-1/2}}$: rewrite as $4x^{-1/2}$ first, then differentiate.
- Differentiating a product by multiplying the derivatives of the two factors, or a fraction by dividing the derivatives. Expand or split first.
- Forgetting that when $n = -\frac{1}{2}$ the new power is $-\frac{3}{2}$ (subtract 1, do not add).

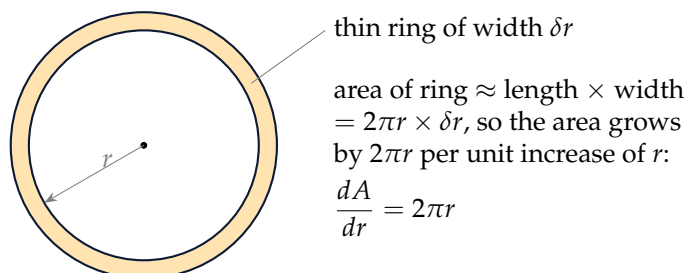
4 Rates of change and the second derivative

The gradient of a graph of y against x tells you how fast y is changing for each unit increase in x . So $\frac{dy}{dx}$ is **the rate of change of y with respect to x** , and its units are "units of y per unit of x ": metres per second, pounds per day, litres per minute.

Here is a distance–time graph for a runner with $s = t^2$ metres after t seconds. The average speed between two times is the gradient of a chord; the speed *at one instant* is the gradient of the tangent, which is the derivative.



Check the idea numerically: between $t = 3$ and $t = 4$ the average speed is $\frac{16 - 9}{1} = 7$ m/s; between $t = 3$ and $t = 3.1$ it is $\frac{9.61 - 9}{0.1} = 6.1$ m/s; and $\frac{ds}{dt} = 2t = 6$ at $t = 3$. **Rates of change from geometry.** The area of a circle is $A = \pi r^2$, so $\frac{dA}{dr} = 2\pi r$, and the number $2\pi r$ is the circumference. That is not a coincidence. Make the radius a little bigger by δr : the extra area is a thin ring around the edge, whose area is about its length (the circumference) times its width δr . So the area grows by about $2\pi r$ for each unit of extra radius.



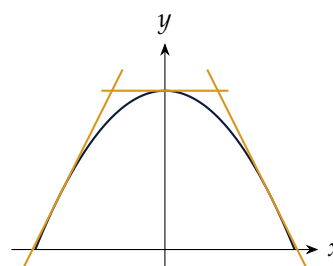
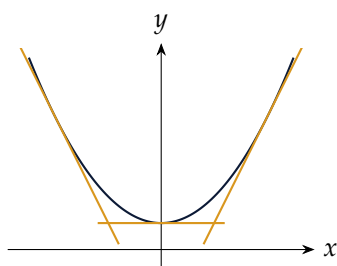
In the same way the volume of a sphere $V = \frac{4}{3}\pi r^3$ has $\frac{dV}{dr} = 4\pi r^2$, the surface area: a thin shell of thickness δr has volume about (surface area) $\times \delta r$.

Worked example: a rate of change of volume

A sphere has radius r cm. Find the rate at which its volume changes with r when $r = 3$, and state the units.

$\frac{dV}{dr} = 4\pi r^2 = 4\pi \times 9 = 36\pi$. The units are cm^3 per cm, so the volume increases by about $36\pi \approx 113 \text{ cm}^3$ for each extra centimetre of radius at that moment.

The second derivative. The derivative is itself a function, so we can differentiate it again. The result is the **second derivative**, written $f''(x)$ or $\frac{d^2y}{dx^2}$. It is the rate of change of the gradient. Where $f'' > 0$ the gradient is increasing (the curve bends upwards, like a bowl); where $f'' < 0$ the gradient is decreasing (it bends downwards, like a dome).



$f''(x) > 0$: tangent gradients $-2, 0, 2$ increase $f''(x) < 0$: tangent gradients $2, 0, -2$ decrease

For motion, if s is displacement, then $\frac{ds}{dt}$ is velocity and $\frac{d^2s}{dt^2}$ is acceleration.

Worked example: velocity and acceleration

A particle has displacement $s = t^3 - 6t^2 + 9t$ metres at time t seconds. Find its velocity and acceleration at $t = 1$ and describe the motion.

$v = \frac{ds}{dt} = 3t^2 - 12t + 9$ and $a = \frac{d^2s}{dt^2} = 6t - 12$. At $t = 1$: $v = 0$ and $a = -6$. The particle is momentarily at rest and its velocity is decreasing, so it is just about to move backwards (towards the start).

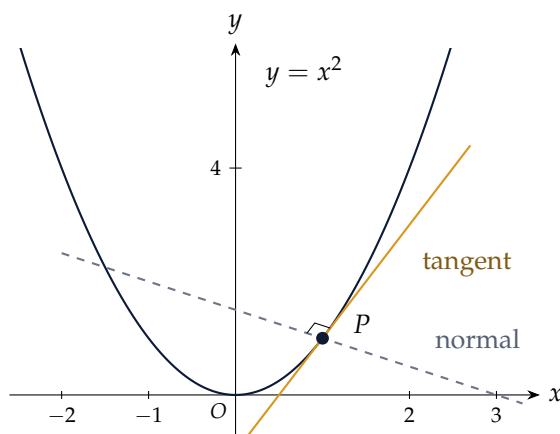
Common mistakes

- Forgetting the units of a rate. $\frac{dV}{dr}$ is volume *per unit of radius*, not a volume.
- Confusing the average rate (a chord gradient) with the instantaneous rate (the derivative).
- Writing $\frac{d^2y}{dx^2}$ as $\left(\frac{dy}{dx}\right)^2$. The second derivative is the derivative of the derivative, not its square.

5 Tangents and normals

The **tangent** to a curve at a point has the same gradient as the curve there. The **normal** is the line through the same point perpendicular to the tangent. We find either with the same three steps:

1. Find the point: put the given x into the *curve* to get y .
2. Find the gradient: put the same x into the *derivative*. This is m for the tangent. For the normal use $-\frac{1}{m}$ (perpendicular lines have gradients that multiply to -1 ; if $m = 0$ the normal is vertical).
3. Write the line using $y - y_1 = m(x - x_1)$.

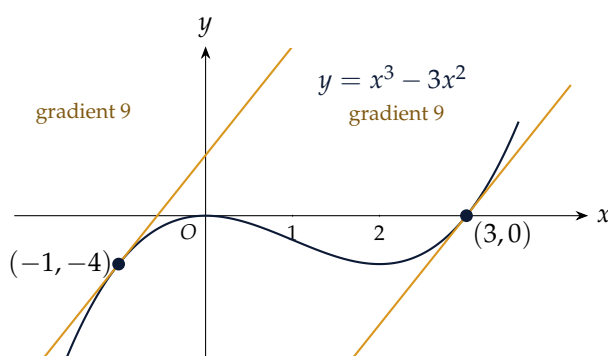


Worked example: a tangent and a normal

Find the tangent and the normal to $y = x^2$ at $P(1, 1)$ (shown above).

$\frac{dy}{dx} = 2x = 2$ at $x = 1$. Tangent: $y - 1 = 2(x - 1)$, so $y = 2x - 1$. The normal has gradient $-\frac{1}{2}$: $y - 1 = -\frac{1}{2}(x - 1)$, so $2y - 2 = -x + 1$ and $x + 2y = 3$.

Where is the tangent parallel to a given line? Parallel lines have equal gradients, so solve $\frac{dy}{dx} = (\text{the given gradient})$. There may be several answers, because a curve can have the same gradient at more than one point.



Worked example: parallel tangents

Find the points on $y = x^3 - 3x^2$ where the tangent is parallel to $y = 9x + 1$.

$\frac{dy}{dx} = 3x^2 - 6x$. Set this equal to 9: $3x^2 - 6x - 9 = 0$, so $x^2 - 2x - 3 = 0$ and $(x - 3)(x + 1) = 0$. The points are $(3, 0)$ and $(-1, -4)$. The tangent at $(3, 0)$ is $y = 9x - 27$ and at $(-1, -4)$ it is $y = 9x + 5$, as drawn above.

Worked example: when is a line a tangent?

For what value of c is the line $y = 4x - c$ a tangent to $y = x^2$?

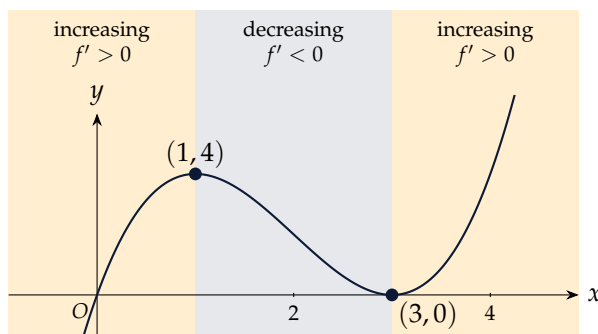
The tangent must have gradient 4, so $2x = 4$ and the point of contact is $x = 2, y = 4$. The line passes through $(2, 4)$: $4 = 8 - c$, so $c = 4$. (Check by algebra: $x^2 = 4x - c$ gives $x^2 - 4x + c = 0$, which has a repeated root when the discriminant $16 - 4c = 0$, again $c = 4$.)

Common mistakes

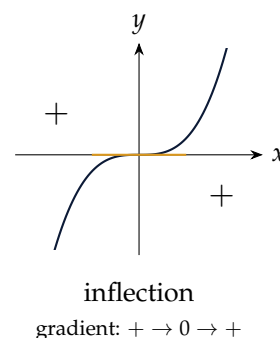
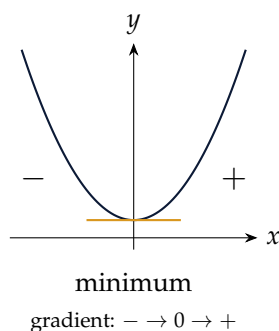
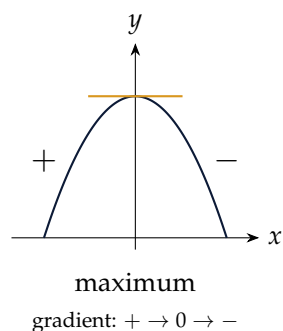
- Using x in the derivative to find the y -coordinate of the point. The y -coordinate comes from the *curve*.
- Using the tangent gradient for the normal. The normal's gradient is the negative reciprocal.
- Stopping after finding one point when asked where a tangent has a given gradient: solve the equation fully.

6 Stationary points and the shape of a curve

A function is **increasing** where $f'(x) > 0$ (the curve climbs as you move right) and **decreasing** where $f'(x) < 0$. A **stationary point** is where $f'(x) = 0$: the tangent is horizontal. Take $y = x^3 - 6x^2 + 9x$, so $\frac{dy}{dx} = 3x^2 - 12x + 9 = 3(x-1)(x-3)$. The gradient is zero at $x = 1$ and $x = 3$, positive for $x < 1$, negative for $1 < x < 3$ and positive again for $x > 3$.



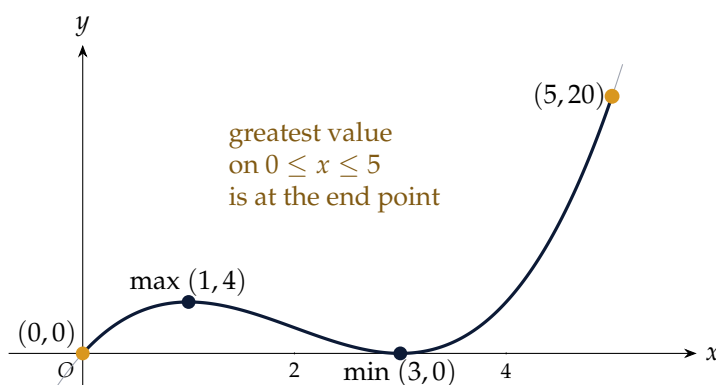
There are three kinds of stationary point. A **maximum** has the gradient changing from $+$ to 0 to $-$ (the curve climbs, levels, then falls). A **minimum** has $-$, 0 , $+$. A **point of inflection** with a horizontal tangent has the *same* sign either side, so the curve flattens for an instant but keeps going the same way.



Two ways to decide which it is.

1. *The sign test (always works).* Check the sign of f' just to the left and right of the point.
2. *The second-derivative test (quicker).* At a stationary point, $f'' > 0$ means a minimum and $f'' < 0$ means a maximum. The reason: f'' is the rate of change of the gradient. If $f'' < 0$ the gradient is decreasing as it passes through 0, so it goes from positive to negative: a maximum. If $f'' = 0$ the test says nothing, so use the sign test.

Local versus greatest. A maximum is only a *local* high point: the highest point nearby. The greatest value of a function on a given interval is found by comparing the local maxima with the values at the two end points.



Worked example: classifying stationary points

Find and classify the stationary points of $y = 2x^3 - 9x^2 + 12x$.

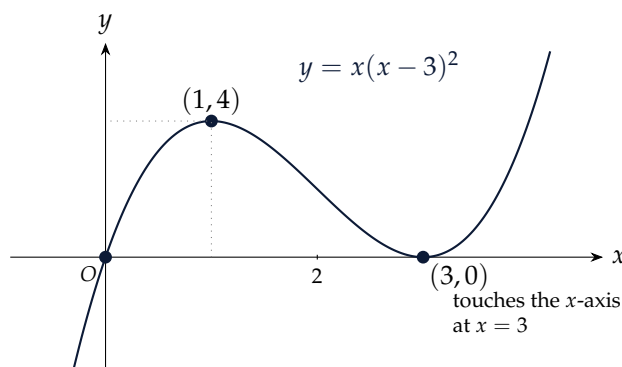
$\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x-1)(x-2)$, so $x = 1$ or $x = 2$. The second derivative is $\frac{d^2y}{dx^2} = 12x - 18$. At $x = 1$ it is $-6 < 0$: a maximum with $y = 2 - 9 + 12 = 5$. At $x = 2$ it is $6 > 0$: a minimum with $y = 16 - 36 + 24 = 4$. The stationary points are the maximum $(1, 5)$ and the minimum $(2, 4)$.

Worked example: when the second derivative is zero

Find and classify the stationary points of $y = x^4 - 4x^3$.

$\frac{dy}{dx} = 4x^3 - 12x^2 = 4x^2(x-3)$, so $x = 0$ or $x = 3$. At $x = 3$: $\frac{d^2y}{dx^2} = 12x^2 - 24x = 36 > 0$, a minimum, with $y = 81 - 108 = -27$. At $x = 0$ the second derivative is 0, which tells us nothing. Use the sign test: $\frac{dy}{dx} = 4x^2(x-3)$ is negative just left of 0 and negative just right of 0 (because $x^2 > 0$ and $x-3 < 0$). The sign does not change, so $(0, 0)$ is a point of inflection.

Sketching with calculus. Combine the intercepts with the stationary points. For $y = x^3 - 6x^2 + 9x = x(x-3)^2$: it crosses the y -axis at the origin; it meets the x -axis at $x = 0$ and at $x = 3$, where the squared factor means it *touches* the axis without crossing; the stationary points are $(1, 4)$ (maximum) and $(3, 0)$ (minimum); and for large positive x it rises, for large negative x it falls.



Worked example: greatest value on an interval

Find the greatest and least values of $y = x^3 - 6x^2 + 9x$ for $0 \leq x \leq 5$.

The candidates are the stationary points inside the interval, $(1, 4)$ and $(3, 0)$, and the end points, where $y(0) = 0$ and $y(5) = 125 - 150 + 45 = 20$. The greatest value is 20 (at the end point $x = 5$) and the least value is 0 (at $x = 0$ and $x = 3$).

Common mistakes

- Giving the x -value when the question asks for the y -value of the stationary point, or the other way round.
- Saying a stationary point is a maximum because f'' is *positive*. Positive means minimum (bowl shape).
- Assuming every stationary point is a maximum or minimum: $y = x^3$ has $f'(0) = 0$ but is rising on both sides.
- Forgetting the end points when asked for the greatest value on an interval.

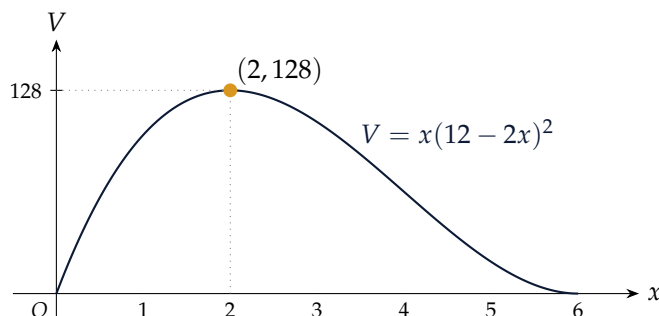
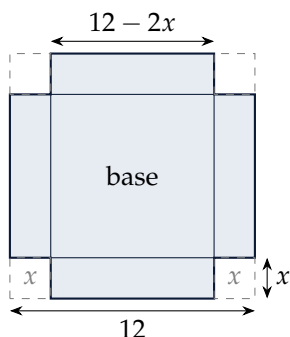
7 Optimisation

Optimisation means finding the greatest or least value of a quantity. The method is always the same.

1. Draw a diagram and name the variables.
2. Write the quantity to be optimised (A , V , cost...) as a function of *one* variable, using the constraint in the question to eliminate the others.
3. Differentiate and solve $\frac{d}{dx}(\text{quantity}) = 0$.

- Show that it is a maximum or minimum (second derivative or sign test), and check the end points if the variable has a limited range.
- Answer the question that was asked: the greatest *area*, not the value of x .

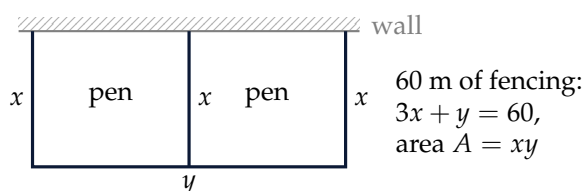
An open box. A 12 cm square sheet has a square of side x cut from each corner, and the sides are folded up.



Worked example: greatest volume of the box

Find the greatest volume.

The base is a square of side $12 - 2x$ and the height is x , so $V = x(12 - 2x)^2$ with $0 < x < 6$. Expanding, $V = 144x - 48x^2 + 4x^3$, so $\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(x^2 - 8x + 12) = 12(x - 2)(x - 6)$. For $0 < x < 6$ the only stationary point is $x = 2$. Here $\frac{d^2V}{dx^2} = 24x - 96 = -48 < 0$, a maximum. $V(2) = 2 \times 8^2 = 128 \text{ cm}^3$. The graph confirms it: V is 0 at both ends of the range.



Worked example: fencing against a wall

A farmer has 60 m of fencing and builds two equal rectangular pens against a straight wall, as in the diagram (no fence is needed along the wall). Find the greatest total area.

The fence has three sections of length x and one of length y , so $3x + y = 60$ and $y = 60 - 3x$. The area is $A = xy = 60x - 3x^2$. Then $\frac{dA}{dx} = 60 - 6x = 0$ gives $x = 10$, with $\frac{d^2A}{dx^2} = -6 < 0$ (maximum). Then $y = 30$ and $A = 10 \times 30 = 300 \text{ m}^2$. (A quadratic could also be done by symmetry: $A = 3x(20 - x)$ is zero at $x = 0$ and $x = 20$, so its maximum is half-way at $x = 10$.)

Worked example: a can of least surface area

A closed cylindrical can must hold a volume of $54\pi \text{ cm}^3$. Find the radius that makes its surface area least, and that least area.

$V = \pi r^2 h = 54\pi$ gives $h = \frac{54}{r^2}$. The surface area is $S = 2\pi r^2 + 2\pi rh = 2\pi r^2 + \frac{108\pi}{r} = 2\pi r^2 + 108\pi r^{-1}$. Then $\frac{dS}{dr} = 4\pi r - 108\pi r^{-2} = 0$ gives $r^3 = 27$, so $r = 3$. The second derivative $\frac{d^2S}{dr^2} = 4\pi + 216\pi r^{-3} > 0$, so it is a minimum. Then $h = 6$ (twice the radius) and $S = 18\pi + 36\pi = 54\pi \text{ cm}^2$.

Common mistakes

- Differentiating while two variables are still present. Use the constraint to get a function of *one* variable first.
- Giving x (or r) as the final answer when the question asks for the greatest area or volume.
- Not checking the nature of the stationary point or the end points: $V = x(12 - 2x)^2$ is zero at both ends, so the

maximum must be in between.

8 Strategy and intuitive checks

Read the question for what it asks. “Find the gradient” needs $\frac{dy}{dx}$ and a substitution; “find the point” needs the curve; “find the greatest value” needs a y (or area, volume) value, not an x . After finding a stationary point, ask: which coordinate does the question want?

Use the graph. If a curve is rising, $f' > 0$; at a smooth top or bottom, $f' = 0$; where the curve bends like a bowl, $f'' > 0$. Use these to eliminate wrong answers before calculating: the gradient of a falling curve cannot be positive.

Distance travelled is not displacement. A particle with displacement s that turns round has travelled further than its final displacement. Split the motion at the times when $\frac{ds}{dt} = 0$ and add the *sizes* of the separate legs.

Worked example: distance travelled

A particle has displacement $s = t^3 - 6t^2 + 9t$ m from O . How far does it travel in the first 4 seconds?

$\frac{ds}{dt} = 3t^2 - 12t + 9 = 3(t-1)(t-3)$, so it turns round at $t = 1$ and $t = 3$. $s(0) = 0$, $s(1) = 4$, $s(3) = 0$ and $s(4) = 64 - 96 + 36 = 4$. The legs are $0 \rightarrow 4$ (distance 4), $4 \rightarrow 0$ (distance 4) and $0 \rightarrow 4$ (distance 4): a total of 12 m, although the final displacement is only 4 m.

Exam tip

In optimisation problems the answer is often a “nice” number. If you get an awkward value, recheck the algebra, especially the constraint equation, before trusting it.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Differentiate $y = 3x^4 - 2x^{-1} + 5\sqrt{x}$.
2. Use the chord from $x = 3$ to $x = 3 + h$ on $y = x^2$ to find the gradient at $x = 3$.
3. Find the gradient of $y = x^2(x - 1)$ at $x = 2$.
4. Find the equation of the tangent and the normal to $y = x^2 - 4x$ at $x = 1$.
5. Find and classify the stationary point of $y = x^2 - 6x + 5$.
6. For which x is $y = x^3 - 12x$ increasing?
7. Two numbers add to 12. Find the greatest possible product.
8. The area of a circle is $A = \pi r^2$. Find the rate of change of A with r when $r = 5$ cm.

Answers to the self-check

1. $12x^3 + 2x^{-2} + \frac{5}{2}x^{-1/2}$.
2. Gradient of chord $= \frac{(3+h)^2 - 9}{h} = 6 + h \rightarrow 6$.
3. $y = x^3 - x^2$, $y' = 3x^2 - 2x$, so $12 - 4 = 8$.
4. $y(1) = -3$, $y' = 2x - 4 = -2$. Tangent: $y + 3 = -2(x - 1)$, i.e. $y = -2x - 1$. Normal gradient $\frac{1}{2}$: $y + 3 = \frac{1}{2}(x - 1)$, i.e. $x - 2y = 7$.
5. $y' = 2x - 6 = 0$ at $x = 3$, $y = -4$; $y'' = 2 > 0$ so a minimum at $(3, -4)$.
6. $y' = 3x^2 - 12 > 0$ when $x < -2$ or $x > 2$.
7. $P = x(12 - x)$, $P' = 12 - 2x = 0$ at $x = 6$: $P = 36$.
8. $\frac{dA}{dr} = 2\pi r = 10\pi \text{ cm}^2$ per cm.

Next step

Heurisco's *ESAT Maths 2: Differentiation Practice Set* has twenty original exam-style questions on this lesson, with diagrams, in order of difficulty, with a full worked solution for each.