

ESAT

Maths 2

Integration

Free lesson notes

Covers

MM7.1 to MM7.6

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

This lesson covers integrating powers of x , definite integrals and the area between a curve and the x -axis, the Fundamental Theorem of Calculus, combining integrals over joined or equal ranges, the trapezium rule with over- and under-estimates, and finding a function from its derivative.

How to use it. Try each worked example first. Then do the self-check, then the practice set. Sketch the curve whenever a question asks for an *area*: a definite integral counts area below the axis as negative, so the sketch tells you whether you need to split the range.

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1 Integrating powers of x

Integration reverses differentiation. For any rational $n \neq -1$,

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C.$$

Integrate term by term, and always rewrite roots and reciprocals as powers first ($\sqrt{x} = x^{1/2}$, $\frac{1}{x^3} = x^{-3}$) and expand brackets. If you are given $\frac{dy}{dx} = f(x)$ and a point on the curve, integrate and use the point to find the constant C .

Worked example: rewriting first

Find $\int \left(\frac{x+1}{\sqrt{x}} \right) dx$.

$\frac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2}$, so the integral is $\frac{2}{3}x^{3/2} + 2x^{1/2} + C$.

Worked example: finding the constant

$\frac{dy}{dx} = 6x^2 - 2$ and the curve passes through $(1, 5)$. Find y when $x = 2$.

$y = 2x^3 - 2x + C$. At $(1, 5)$: $2 - 2 + C = 5$, so $C = 5$. At $x = 2$: $y = 16 - 4 + 5 = 17$.

Common mistakes

- Using $n = -1$: $\int x^{-1} dx$ is not $\frac{x^0}{0}$. (It is not part of this specification.)
- Forgetting $+C$ when it is needed, or forgetting to use the given point to find it.
- Integrating $\frac{x+1}{\sqrt{x}}$ as $\frac{\int(x+1)}{\int\sqrt{x}}$: there is no quotient rule for integrals.

2 Definite integrals and area

A definite integral is evaluated with limits: $\int_a^b f(x) dx = F(b) - F(a)$, where F is any antiderivative. Geometrically it is the signed area between the curve and the x -axis: area *below* the axis counts as negative. To find a *total* area where the curve crosses the axis, integrate each part separately and add the absolute values. The area between two curves $y = f(x)$ (above) and $y = g(x)$ (below) is $\int_a^b (f(x) - g(x)) dx$.

Worked example: area under a curve

Find the area between $y = 4x - x^2$ and the x -axis.

The curve meets the axis at $x = 0$ and $x = 4$ and is above it between them. Area $= \int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}$.

Worked example: area where the curve crosses the axis

Find the total area enclosed between $y = x^3 - x$ and the x -axis, from $x = -1$ to $x = 1$.

The integral over $[-1, 1]$ is 0, because the two halves cancel. The areas are equal, so the total is $2 \int_0^1 (x - x^3) dx = 2 \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{2}$.

Common mistakes

- Treating a definite integral as the area when part of the region lies below the axis: it gives the *signed* area, and the parts may cancel.
- Subtracting in the wrong order: it is $F(b) - F(a)$, upper limit first.
- Using the integral of the lower curve minus the upper curve: you will get a negative area.

3 Fundamental Theorem and combining integrals

The Fundamental Theorem of Calculus links the two operations: if $F(x) = \int_a^x f(t) dt$ then $F'(x) = f(x)$. Integrals over adjacent ranges add, and reversing the limits changes the sign:

$$\int_a^c f = \int_a^b f + \int_b^c f, \quad \int_b^a f = - \int_a^b f, \quad \int (pf + qg) = p \int f + q \int g.$$

Worked example: adjacent ranges

$\int_0^5 f(x) dx = 7$ and $\int_3^5 f(x) dx = 2$. Find $\int_0^3 (2f(x) + 1) dx$.
 $\int_0^3 f = 7 - 2 = 5$, so $\int_0^3 (2f + 1) = 2(5) + 3 = 13$.

Worked example: differentiating an integral

$F(x) = \int_2^x (3t^2 - 1) dt$. Find $F'(2)$.
 $F'(x) = 3x^2 - 1$, so $F'(2) = 11$.

Common mistakes

- Differentiating $\int_a^x f(t) dt$ by integrating again: the result is simply $f(x)$.
- Forgetting that the constant 1 integrates to x , giving 3 over a range of width 3.

4 The trapezium rule

When a function cannot be integrated easily, approximate the area with trapezia. With n strips of width $h = \frac{b-a}{n}$ and ordinates $y_0, y_1, \dots, y_n$:

$$\int_a^b f(x) dx \approx \frac{h}{2} [y_0 + y_n + 2(y_1 + \dots + y_{n-1})].$$

The tops of the trapezia are chords. If the curve is *convex* (curving upwards, $f'' > 0$) the chords lie above the curve and the rule *over-estimates*; if it is *concave* it under-estimates.

Worked example: using the rule

Use the trapezium rule with 4 strips to estimate $\int_0^2 x^2 dx$.

$h = 0.5$ and $y = 0, 0.25, 1, 2.25, 4$. Estimate $= \frac{0.5}{2} [0 + 4 + 2(0.25 + 1 + 2.25)] = 0.25 \times 11 = 2.75$. The exact value is $\frac{8}{3} \approx 2.67$, so it over-estimates, as the curve is convex.

Worked example: a reciprocal

Use two strips to estimate $\int_1^3 \frac{1}{x} dx$.

$h = 1$ and $y = 1, \frac{1}{2}, \frac{1}{3}$. Estimate $= \frac{1}{2} \left[1 + \frac{1}{3} + 2 \cdot \frac{1}{2} \right] = \frac{1}{2} \cdot \frac{7}{3} = \frac{7}{6}$.

Common mistakes

- Using the number of ordinates as the number of strips: 4 strips use 5 ordinates.
- Doubling the end ordinates, or forgetting to double the middle ones.
- Saying the rule over-estimates for every curve: it depends on the curvature.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Find $\int (6x^2 - 4x + 3) dx$.
2. Evaluate $\int_1^4 \sqrt{x} dx$.
3. Find the area between $y = x^2$ and $y = 2x$.
4. Given $\int_1^3 f = 4$ and $\int_3^6 f = 5$, find $\int_1^6 f$ and $\int_6^1 f$.
5. Find $a > 0$ such that $\int_0^a 2x dx = 16$.
6. $f'(x) = 3x^2$ and $f(0) = 2$. Find $f(2)$.

Answers to the self-check

1. $2x^3 - 2x^2 + 3x + C$.
2. $\left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8 - 1) = \frac{14}{3}$.
3. They meet at $x = 0, 2$. $\int_0^2 (2x - x^2)dx = 4 - \frac{8}{3} = \frac{4}{3}$.
4. 9 and -9 .
5. $a^2 = 16$, so $a = 4$.
6. $f(x) = x^3 + 2$, so $f(2) = 10$.

Next step

Heurisco's *ESAT Maths 2: Integration Practice Set* has twenty original exam-style questions on this lesson, in order of difficulty, with a full worked solution for each.