

ESAT

Maths 2

Graphs of functions

Free lesson notes

Covers

MM8.1 to MM8.7

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

This lesson covers sketching the standard curves (lines, quadratics, cubics, trigonometric, exponential, logarithmic, square-root and modulus graphs), the four basic transformations and their combinations, the roles of a , b and c in $a(x + b)^2 + c$, stationary points and axis intersections, and how the graph shows the number of solutions of an equation.

How to use it. Try each worked example first. Then do the self-check, then the practice set. For almost every question here, a quick sketch with the key features labelled (intercepts, turning points, asymptotes) is faster and safer than algebra.

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1 Sketching the standard curves

For each curve, know the shape and the features to label:

- **Quadratic** $y = ax^2 + bx + c$: a parabola, U-shaped if $a > 0$; y -intercept c ; roots from the factors or the discriminant; vertex from completing the square.
- **Cubic**: crosses the axis once, twice (touching at one) or three times; y -intercept from the constant term.
- $y = \sin x$, $\cos x$, $\tan x$: periodic; $\sin$ and $\cos$ lie between -1 and 1 ; $\tan$ has vertical asymptotes at $\pm \frac{\pi}{2}, \dots$
- $y = a^x$: positive, passes through $(0, 1)$, asymptote $y = 0$. $y = \log_a x$: defined for $x > 0$, passes through $(1, 0)$, asymptote $x = 0$.
- $y = \sqrt{x}$: starts at the origin and rises with decreasing gradient. $y = |f(x)|$: the parts of $y = f(x)$ below the axis are reflected above it.

Worked example: a modulus graph

Solve $|2x - 6| = 4$.

Either $2x - 6 = 4$ giving $x = 5$, or $2x - 6 = -4$ giving $x = 1$. The V-shaped graph of $y = |2x - 6|$ meets the line $y = 4$ twice.

Worked example: a square-root graph

Where does $y = 3 - \sqrt{x + 4}$ cross the x -axis?

$\sqrt{x + 4} = 3$, so $x + 4 = 9$ and $x = 5$.

Common mistakes

- Thinking $y = a^x$ can reach 0 or go negative: it has an asymptote at $y = 0$ but never touches it.
- Drawing $\tan x$ continuous across its asymptotes.
- Forgetting that $|f(x)|$ never goes below the x -axis.

2 Transformations

Starting from $y = f(x)$, with a a positive or negative number:

$$y = f(x) + a \quad \text{translate } a \text{ up (vector } \begin{pmatrix} 0 \\ a \end{pmatrix})$$

$$y = f(x + a) \quad \text{translate } a \text{ to the left (vector } \begin{pmatrix} -a \\ 0 \end{pmatrix})$$

$$y = af(x) \quad \text{stretch vertically, scale factor } a \text{ (and reflect in the } x\text{-axis if } a < 0)$$

$$y = f(ax) \quad \text{stretch horizontally, scale factor } \frac{1}{a} \text{ (and reflect in the } y\text{-axis if } a < 0)$$

Changes *inside* the brackets (to x) act on the x -coordinate and do the *opposite* of what they look like; changes *outside* act on the y -coordinate as written. For a combination, apply them in the order a calculation would: for $y = 2f(x - 1) + 3$, shift right 1, stretch by 2, then shift up 3.

Worked example: a point under transformation

$y = f(x)$ has a maximum at $(2, 5)$. Where is the maximum of $y = 2f(x - 1) + 3$?
 $x: 2 \rightarrow 2 + 1 = 3$. $y: 5 \rightarrow 2(5) + 3 = 13$. The maximum is at $(3, 13)$.

Worked example: a horizontal stretch

$y = f(x)$ passes through $(4, 6)$. Which point lies on $y = f(\frac{x}{2})$?
 The x -coordinates are multiplied by 2: $(8, 6)$.

Common mistakes

- Moving $f(x + 3)$ to the right. It moves left.
- Halving the x -coordinates for $f(\frac{x}{2})$: the stretch factor is 2.
- Applying $2f(x - 1) + 3$ in the wrong order, e.g. adding 3 before doubling.

3 The quadratic $a(x + b)^2 + c$

Completing the square writes $y = ax^2 + px + q$ as $a(x + b)^2 + c$. The graph is a translation of $y = ax^2$ by the vector $(\begin{smallmatrix} -b \\ c \end{smallmatrix})$, with vertex $(-b, c)$, minimum value c if $a > 0$ (maximum if $a < 0$) and axis of symmetry $x = -b$. The sign and size of a control the direction and steepness; c is the turning-point value. To find a parameter from the vertex and another point, substitute.

Worked example: finding a

$y = a(x - 2)^2 - 4$ passes through $(0, 8)$. Find a .
 $a(4) - 4 = 8$, so $4a = 12$ and $a = 3$.

Worked example: translating a parabola

$y = x^2 - 4x + 1$ is translated by $(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix})$. Find the y -intercept of the new curve.
 New curve: $y = (x - 3)^2 - 4(x - 3) + 1 - 2 = x^2 - 10x + 20$. At $x = 0$: $y = 20$.

Common mistakes

- Reading the vertex of $y = (x + 3)^2 - 5$ as $(3, -5)$: the x -coordinate is -3 .
- Substituting $x = 0$ into the original equation after a translation.

4 Intersections and the number of solutions

Solutions of $f(x) = g(x)$ are the x -coordinates of the points where the graphs meet. To count them without solving, sketch both curves. To find where $f(x) = k$ has three solutions, find the stationary values of f : a horizontal line crosses a cubic three times exactly when it lies between the local minimum and local maximum values. A line is a tangent where the equation has a repeated root (discriminant = 0).

Worked example: counting roots

How many real solutions does $x^3 - 3x = 1$ have?
 $y = x^3 - 3x$ has a local maximum 2 at $x = -1$ and a local minimum -2 at $x = 1$. The line $y = 1$ lies between -2 and 2, so it crosses three times.

Worked example: a tangent line

For which negative m is the line $y = mx$ tangent to $y = x^2 - 4x + 4$?
 $x^2 - (4 + m)x + 4 = 0$ needs $\Delta = (4 + m)^2 - 16 = 0$, so $m = 0$ or $m = -8$. The negative value is $m = -8$.

Common mistakes

- Counting intersections from one sketch without checking the stationary values: a curve may just miss a line.
- Forgetting negative solutions, e.g. of $2^x = x^2$ (there is one for $x < 0$).

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. State the asymptote and the y -intercept of $y = 2^x - 3$.
2. $y = f(x)$ has a root at $x = 4$. Where is the root of $y = f(x + 3)$?
3. Find the vertex of $y = 2(x + 1)^2 - 5$.
4. Find the number of real roots of $x^4 - 5x^2 + 4 = 0$.
5. Solve $|x - 3| = 2x$.
6. The curve $y = x^3 - 3x^2 + k$ has a local minimum on the x -axis. Find k .

Answers to the self-check

1. Asymptote $y = -3$; y -intercept -2 .
2. $x = 1$.
3. $(-1, -5)$.
4. $(x^2 - 1)(x^2 - 4) = 0$: four roots $\pm 1, \pm 2$.
5. $x - 3 = 2x$ gives $x = -3$ (rejected, since $|\cdot| \geq 0$ needs $2x \geq 0$); $x - 3 = -2x$ gives $x = 1$.
6. Minimum at $x = 2$: $8 - 12 + k = 0$, so $k = 4$.

Next step

Heurisco's *ESAT Maths 2: Graphs of Functions Practice Set* has twenty original exam-style questions on this lesson, in order of difficulty, with a full worked solution for each.