

ESAT

Maths 2

Applied reasoning and modelling

Free lesson notes

Covers

Applies MM1 to MM8

Format

Lesson, worked examples, self-check

Price

Free

About this lesson

Many Maths 2 questions do not say “use differentiation” or “use a geometric series”. They describe a situation (a ball thrown into the air, a bacteria culture, a Ferris wheel, two runners on a track) and leave you to decide which piece of mathematics fits. This lesson is about that decision. It shows the standard *models* that appear again and again, explains why each one works, and practises turning words into equations.

You will meet seven situations: motion graphs (where integration and differentiation have a physical meaning), quadratic models, optimisation with calculus, exponential growth and decay, sequences and series in context, circular motion and radians, and a closing section of strategy. Every formula is derived or explained as it is introduced, so you can rebuild it under exam pressure if you forget it.

How to use it. Try each worked example yourself before reading the solution. After each section, ask: *what feature of the situation told me which model to use?* That habit is the real skill the test rewards. Then do the self-check and the practice set. No calculator is needed or allowed in the ESAT, so all numbers here are chosen to work out exactly.

Heurisco is an independent practice provider and is not affiliated with, endorsed by, or connected to UAT-UK, Pearson VUE, the UCAT Consortium, or any university. All questions are original.

1 Turning words into mathematics

A reliable routine for any applied question:

1. **Draw a diagram** and mark every given quantity with its units.
2. **Name the unknown** with a letter and say what it measures (“let v be the speed in km/h”).
3. **Write one equation** that says two expressions for the same quantity are equal. Usually that is a relationship like distance = speed $\times$ time, or a total (“fencing used = 40”).
4. **Solve**, then **check** the answer against the story: positive lengths, sensible sizes, the right units, and whether you answered the question that was asked.

Equations from the real world often turn out to be quadratic, because two quantities that both depend on the unknown are multiplied (speed and time, length and width). When you get a quadratic, there are usually two roots and the story rejects one of them.

Worked example: a speed problem that gives a quadratic

A train travels 90 km at a steady speed v km/h. If its speed were 15 km/h greater, the journey would take 30 minutes ($\frac{1}{2}$ hour) less. Find v .

Time = $\frac{\text{distance}}{\text{speed}}$. The two times differ by $\frac{1}{2}$ hour: $\frac{90}{v} - \frac{90}{v+15} = \frac{1}{2}$. Multiply through by $2v(v+15)$: $180(v+15) - 180v = v(v+15)$, so $2700 = v^2 + 15v$ and $v^2 + 15v - 2700 = 0$, i.e. $(v-45)(v+60) = 0$. A speed cannot be negative, so $v = 45$ km/h. Check: $90/45 = 2$ h and $90/60 = 1.5$ h, a difference of 0.5 h.

Common mistakes

- Mixing units: minutes with hours, metres with kilometres. Convert before writing the equation.
- Keeping both roots of a quadratic. Check each against the story: speeds, times and lengths are positive.
- Answering a different question: the equation may give v but the question asks for the time, or the distance.

2 Motion graphs: gradients and areas

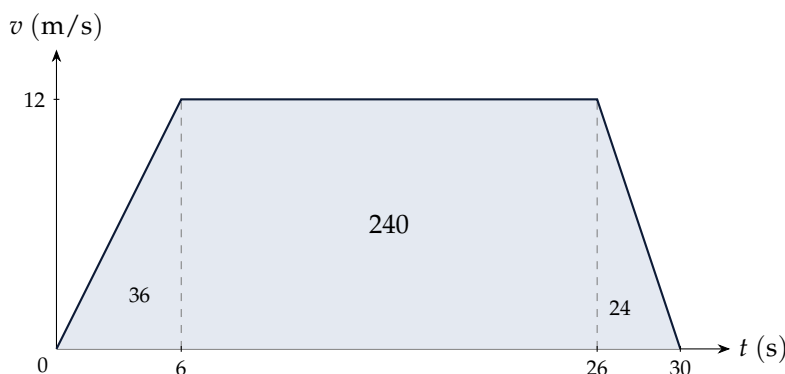
For an object moving in a straight line, three graphs tell the story, and each one links to calculus.

- On a **distance–time graph** the *gradient* is the speed (distance covered per second). Differentiating distance with respect

to time gives speed: $v = \frac{dx}{dt}$.

- On a **speed–time graph** the *gradient* is the acceleration (change of speed per second), $a = \frac{dv}{dt}$, and the *area under the graph* is the distance travelled, because a thin strip of width dt and height v has area $v dt$, the distance covered in that instant. Adding the strips is integration: $x = \int v dt$.

When the speed–time graph is made of straight lines, the area splits into rectangles (base $\times$ height) and triangles ($\frac{1}{2} \times$ base $\times$ height), and no integration is needed.



Worked example: area under a speed–time graph

A cyclist speeds up uniformly from rest to 12 m/s in 6 s, rides at 12 m/s until $t = 26$ s and then slows uniformly to rest at $t = 30$ s. How far does she travel?

Split the area into a triangle, a rectangle and a triangle (see the diagram): $\frac{1}{2}(6)(12) = 36$, then $(26 - 6) \times 12 = 240$, then $\frac{1}{2}(30 - 26)(12) = 24$. The total distance is $36 + 240 + 24 = 300$ m. The *average* speed is $\frac{300}{30} = 10$ m/s.

Worked example: from a formula for velocity

A particle has velocity $v = 6t - t^2$ m/s for $0 \leq t \leq 6$. How far does it travel in these 6 seconds?

The velocity is never negative on this interval ($6t - t^2 = t(6 - t) \geq 0$), so the distance equals the displacement $\int_0^6 (6t - t^2) dt = \left[3t^2 - \frac{t^3}{3}\right]_0^6 = 108 - 72 = 36$ m. (If the velocity changed sign, the object would turn round, and you would have to integrate each part separately and add the positive sizes.)

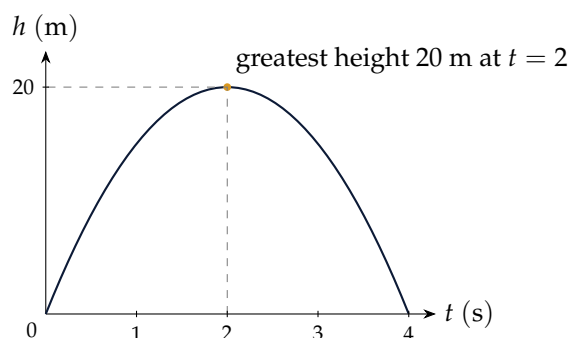
Common mistakes

- Reading the gradient of a speed–time graph as speed: it is acceleration. The *height* is speed.
- Using the formula for a triangle when the shape is a trapezium, or forgetting the $\frac{1}{2}$.
- Forgetting that the horizontal axis is in seconds when the question asks for a time in minutes.

3 Quadratic models

A quantity that rises and then falls, such as the height of a thrown ball, is modelled by a quadratic $h = at^2 + bt + c$ with $a < 0$: its graph is an upside-down parabola. Three facts answer most questions.

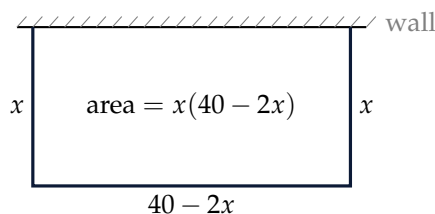
- When is h a given value?** Set the quadratic equal to that value and solve. There can be two answers (once on the way up, once on the way down).
- Greatest value.** The parabola is symmetrical, so the maximum is half-way between the two roots, or equivalently at $t = -\frac{b}{2a}$. Completing the square gives it directly: $a(t + p)^2 + q$ has maximum q at $t = -p$.
- Total time.** The roots of $h = 0$ give the launch and landing times.



Worked example: a thrown ball

The height of a ball is $h = 20t - 5t^2$ metres at t seconds. Find when it lands, its greatest height, and when it is at 15 m.

Landing: $20t - 5t^2 = 5t(4 - t) = 0$ gives $t = 0$ (launch) and $t = 4$ (landing). The maximum is half-way, at $t = 2$: $h = 40 - 20 = 20$ m. At 15 m: $20t - 5t^2 = 15$, so $t^2 - 4t + 3 = 0$, $(t - 1)(t - 3) = 0$, giving $t = 1$ (rising) and $t = 3$ (falling). Notice 1 and 3 are symmetrical about $t = 2$.



Worked example: greatest area with a fixed length of fence

A rectangular pen is built against a straight wall, with fencing on the other three sides. There are 40 m of fencing. What is the greatest possible area?

Let each of the two sides at right angles to the wall be x m. The side parallel to the wall is $40 - 2x$. The area is $A = x(40 - 2x) = 40x - 2x^2$. Complete the square: $A = -2(x^2 - 20x) = -2(x - 10)^2 + 200$. The greatest area is 200 m^2 , when $x = 10$ (so the side along the wall is 20 m, twice the other).

Common mistakes

- Giving the time of greatest height as the answer when the question asks for the height itself (or the other way round).
- Forgetting that there is no fence along the wall: the equation is $2x + y = 40$, not $2x + 2y = 40$.
- Taking the negative root for a time or length.

4 Optimising with calculus

When a quantity is not a simple quadratic, calculus finds its largest or smallest value. The idea: at the top of a hill (or the bottom of a valley) the slope is zero. So:

1. Write the quantity to be maximised (or minimised) as a function of *one* variable, using the constraint to eliminate the others.
2. Differentiate and solve $\frac{df}{dx} = 0$.
3. Decide whether you have a maximum or a minimum: look at the sign of the gradient either side of the point, or use the second derivative ($\frac{d^2f}{dx^2} < 0$ means a maximum, > 0 a minimum). Also check the ends of the allowed range.
4. Answer the question asked: the largest *value*, or the value of x at which it occurs.

Worked example: the biggest open box

A square sheet of card with side 18 cm has a square of side x cm cut from each corner. The flaps are folded up to make an open box. Find the largest possible volume.

The base has side $18 - 2x$ and the height is x , so $V = x(18 - 2x)^2 = 4x^3 - 72x^2 + 324x$ for $0 < x < 9$. Then $\frac{dV}{dx} = 12x^2 - 144x + 324 = 12(x^2 - 12x + 27) = 12(x - 3)(x - 9)$. The only stationary point in the range is $x = 3$. For x slightly below 3 the gradient is positive and slightly above it is negative, so it is a maximum. $V = 3(12)^2 = 432 \text{ cm}^3$.

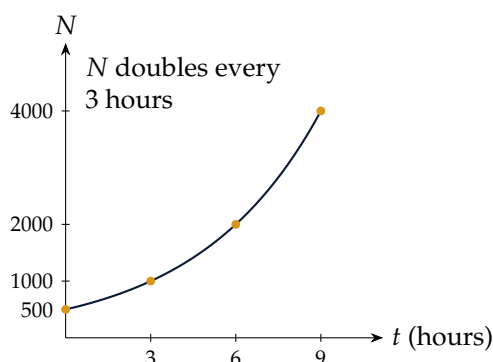
Common mistakes

- Finding the stationary point and stopping there. Substitute back to get the quantity the question asks about.
- Leaving two variables in the function. Use the constraint (fixed length, fixed area) to write everything in terms of one.
- Not checking whether the stationary point is a maximum or a minimum.

5 Exponential growth and decay

If a quantity is multiplied by the same factor in each equal time interval, it follows an exponential model. Two things to recognise:

- A **percentage change** is a multiplier. Increasing by 5% means multiplying by 1.05; decreasing by 20% means multiplying by 0.8. After n periods the quantity is multiplied by the factor n times: $N = N_0 \times (\text{factor})^n$.
- A **doubling time** T means the quantity doubles every T hours, so after t hours it has doubled t/T times: $N = N_0 \cdot 2^{t/T}$. Similarly a **half-life** T gives $N = N_0 \left(\frac{1}{2}\right)^{t/T}$, and a tripling time gives $N_0 \cdot 3^{t/T}$.



To find *when* the quantity reaches a given size, write both sides as powers of the same base and equate the indices. When the numbers do not match neatly, take logarithms: if $a^x = b$ then $x = \log_a b$, because the logarithm is by definition the power to which the base must be raised. Useful laws: $\log_a(xy) = \log_a x + \log_a y$ and $\log_a(x^k) = k \log_a x$.

Worked example: a culture that triples

A culture of 100 cells triples every 2 hours. When are there 8100 cells?

$N = 100 \cdot 3^{t/2}$. Setting $100 \cdot 3^{t/2} = 8100$ gives $3^{t/2} = 81 = 3^4$. Equate the indices: $t/2 = 4$, so $t = 8$ hours. (Four triplings, each taking 2 hours.)

Worked example: value falling by a percentage

A machine worth £20 000 loses 20% of its value each year. What is it worth after 3 years?

Each year the value is multiplied by 0.8. After 3 years it is $20\,000 \times 0.8^3 = 20\,000 \times 0.512 = £10\,240$. (It is not $20\,000 - 3 \times 0.2 \times 20\,000 = £8000$: each year's loss is 20% of the *current* value, which gets smaller.)

Common mistakes

- Treating a percentage change as the same amount each year (a linear change). Repeated percentages multiply.
- Using 0.2 instead of 0.8 as the factor for a 20% fall (or 0.05 instead of 1.05 for a 5% rise).
- Giving t/T (the number of doublings) as the answer when the question asks for the time t .

6 Sequences and series in context

Arithmetic sequences add the same *common difference* d each step. The n th term is $a + (n - 1)d$, where a is the first term (the $n - 1$ is because the first term has had no differences added). The sum of the first n terms is

$$S_n = \frac{n}{2}(2a + (n - 1)d) = \frac{n}{2}(\text{first} + \text{last}).$$

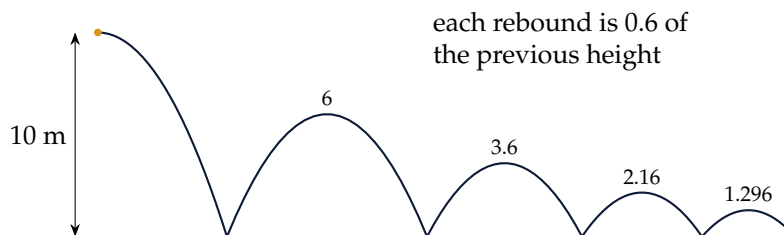
Why? Write the sum forwards and backwards and add: each of the n pairs has the same total (first + last), so twice the sum is $n(\text{first} + \text{last})$.

Geometric sequences multiply by the same *common ratio* r each step: terms $a, ar, ar^2, \dots$ with n th term ar^{n-1} . For the sum, let $S_n = a + ar + \dots + ar^{n-1}$. Multiply by r to get $rS_n = ar + ar^2 + \dots + ar^n$. Subtracting, almost everything cancels: $S_n - rS_n = a - ar^n$, so

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

If $|r| < 1$ then r^n shrinks towards 0 as n grows, so the series settles at a limit, the *sum to infinity*:

$$S_\infty = \frac{a}{1 - r} \quad (|r| < 1).$$



Worked example: a bouncing ball

A ball is dropped from 10 m. After each bounce it rises to 0.6 of its previous height. Find the total distance it travels before coming to rest (in theory).

It first falls 10 m. Then it rises 6 m and falls 6 m, rises 3.6 and falls 3.6, and so on. After the first drop, each height appears twice (up and down), so the extra distance is $2(6 + 3.6 + 2.16 + \dots)$. The inner series is geometric with $a = 6$ and $r = 0.6$: $S_\infty = \frac{6}{1 - 0.6} = 15$. The total is $10 + 2 \times 15 = 40$ m.

Worked example: saving each week

A child saves £5 in week 1 and £3 more each week than the week before. How much has been saved after 20 weeks? An arithmetic sequence with $a = 5, d = 3, n = 20$. The last term is $5 + 19 \times 3 = 62$, and $S_{20} = \frac{20}{2}(5 + 62) = 670$. So £670.

Counting choices (the binomial coefficient). Many applied questions ask “in how many ways can I choose k items from n ?” when the order does not matter. If order did matter, there would be $n(n - 1) \dots (n - k + 1)$ ordered selections (choose a first item in n ways, a second in $n - 1$ ways, and so on). Each unordered group of k items has been counted $k! = k(k - 1) \dots 1$ times, once for each way of ordering it, so the number of groups is

$$\binom{n}{k} = \frac{n!}{k!(n - k)!}$$

read “ n choose k ”, where the factorial $n! = n(n - 1) \dots 2 \cdot 1$. For example $\binom{6}{3} = \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 20$. This is exactly what you need for probability with repeated trials: a fair coin tossed 6 times has $2^6 = 64$ equally likely sequences, and the number with exactly 3 heads is the number of ways to choose *which* 3 of the 6 tosses show heads, $\binom{6}{3} = 20$, so the probability is $\frac{20}{64} = \frac{5}{16}$.

Common mistakes

- Using the formula for the sum to infinity when $|r| \geq 1$. The series only settles down if the ratio is between -1 and 1 .
- Counting the ball's first drop twice (it only falls, it does not bounce up first) or counting every later height only once (each is passed going up *and* going down).
- Off-by-one errors: the n th term is $a + (n - 1)d$, not $a + nd$.

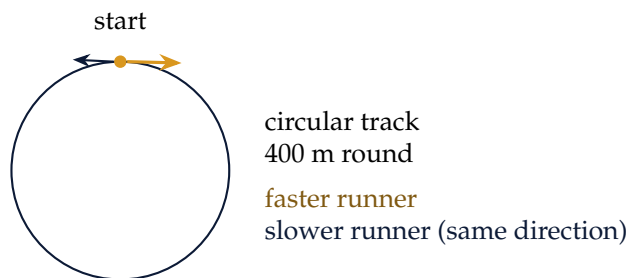
7 Angles, circles and periodic motion

Radians. One *radian* is the angle at the centre of a circle subtended by an arc whose length equals the radius. A full turn is 2π radians ($= 360^\circ$), so π radians $= 180^\circ$. From this: an angle θ (in radians) takes up the fraction $\frac{\theta}{2\pi}$ of a full turn, so

$$\text{arc length} = \frac{\theta}{2\pi} \times 2\pi r = r\theta, \quad \text{sector area} = \frac{\theta}{2\pi} \times \pi r^2 = \frac{1}{2}r^2\theta.$$

(These formulas only work with θ in radians.)

Going round a circle. A runner who completes a lap in time T turns at a steady $\frac{360^\circ}{T}$ per unit time, or equivalently covers $\frac{\text{circumference}}{T}$ of track. When two runners are on the same circular track, work with the *relative* speed, exactly as with the trains in earlier lessons: running in the same direction, the faster gains $(v_1 - v_2)$ per second, so it laps the slower runner each time it has gained one full lap; in opposite directions they close at $v_1 + v_2$, and meet each time their combined distance is one full lap.



Worked example: runners on a track

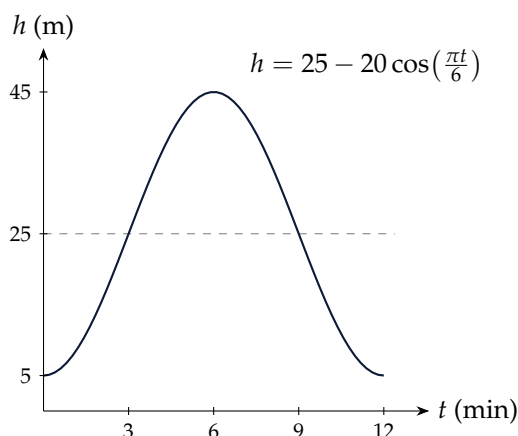
Two runners start together on a 400 m circular track, running in the same direction at 5 m/s and 3 m/s. How often does the faster pass the slower?

The faster runner gains 2 m every second. To gain a full lap of 400 m takes $\frac{400}{2} = 200$ s. So the faster runner draws level every 200 s: at 200, 400, 600, ... seconds. (Running in *opposite* directions they would meet every $\frac{400}{5 + 3} = 50$ s.)

Worked example: the angle between clock hands, and the cosine rule

At 4:00 the minute hand (10 cm) and hour hand (6 cm) of a clock are at an angle of 120° ($4 \text{ hours} \times 30^\circ$). How far apart are their tips?

The two hands and the line joining their tips form a triangle with two known sides and the angle between them. The cosine rule gives $d^2 = 10^2 + 6^2 - 2(10)(6) \cos 120^\circ = 136 - 120(-\frac{1}{2}) = 196$, so $d = 14$ cm.



Worked example: a Ferris wheel

A Ferris wheel of radius 20 m turns once every 12 minutes. Its lowest point is 5 m above the ground, so its centre is 25 m high. A rider starts at the lowest point. The height after t minutes is $h = 25 - 20 \cos\left(\frac{\pi t}{6}\right)$, because the angle turned is $\frac{2\pi}{12}t = \frac{\pi t}{6}$ radians, and at angle 0 the rider is 20 m *below* the centre. When is the rider first 35 m high?

Set $25 - 20 \cos \theta = 35$, so $\cos \theta = -\frac{1}{2}$ and the first solution is $\theta = \frac{2\pi}{3}$. Then $\frac{\pi t}{6} = \frac{2\pi}{3}$ gives $t = 4$ minutes. (Geometrically: 35 m is 10 m above the centre, half the radius, which happens 120° round from the bottom, a third of a turn, i.e. a third of 12 minutes.)

Common mistakes

- Using degrees in $s = r\theta$ or $A = \frac{1}{2}r^2\theta$. These need radians.
- Adding the speeds for runners going the same way (or subtracting them for runners going opposite ways).
- Counting the starting moment as a “pass”. Read the question to see whether it should be included.

8 Strategy for applied questions

- **Look for the structure first.** Is something being multiplied by the same factor repeatedly (exponential or geometric)? Added by the same amount (arithmetic)? Does it rise then fall (quadratic, calculus)? Is the area under a graph a quantity (integration)? Does it go round a circle (angle, periodic)?
- **Use intuition to estimate, then algebra to be exact.** For the runners on a track, you can predict the answer is between a few and a few dozen before calculating.
- **Test extreme cases.** A formula should give sensible answers when the quantity is 0 or very large.
- **Nice numbers are a clue.** ESAT questions are designed to be done without a calculator. If your working produces an ugly number, look for a slip.

Exam tip

Write the units next to every number in your working. If the units on both sides of an equation do not match, you have made a mistake before you have finished the algebra.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. A train travels 60 km at a steady speed. If it were 10 km/h faster, it would take 20 minutes less. Show that its speed satisfies $v^2 + 10v - 1800 = 0$ and find v .
2. A car's speed rises uniformly from 0 to 15 m/s in 5 s, stays constant for 10 s, then falls uniformly to 0 in 5 s. Find the distance.
3. The height of a stone is $h = 30t - 5t^2$. Find its greatest height.
4. A rectangle has perimeter 60 cm. What is its greatest area?
5. A population of 400 halves every 5 years. How many years until it is 25?
6. A ball is dropped from 16 m and rebounds to $\frac{3}{4}$ of its previous height each time. Find the total distance travelled.
7. A sector has radius 6 cm and angle $\frac{\pi}{3}$ radians. Find its arc length and area.
8. How many ways are there to choose 2 different desserts from a menu of 7?

Answers to the self-check

1. $\frac{60}{v} - \frac{60}{v+10} = \frac{1}{3}$ gives $1800 = v(v+10)$, so $v^2 + 10v - 1800 = 0$, i.e. $(v-40)(v+45) = 0$ and $v = 40$ km/h.
2. $\frac{1}{2}(5)(15) + 10(15) + \frac{1}{2}(5)(15) = 37.5 + 150 + 37.5 = 225$ m.
3. Roots $t = 0, 6$; maximum at $t = 3$: $h = 90 - 45 = 45$ m.
4. With sides x and $30 - x$: $A = x(30 - x) = 225 - (x - 15)^2$, greatest value 225 cm^2 (a square).
5. $400 \cdot (\frac{1}{2})^{t/5} = 25$ gives $(\frac{1}{2})^{t/5} = \frac{1}{16} = (\frac{1}{2})^4$, so $t = 20$ years.
6. $16 + 2 \times \frac{12}{1 - \frac{3}{4}} = 16 + 96 = 112$ m.
7. Arc $= 6 \times \frac{\pi}{3} = 2\pi$ cm; area $= \frac{1}{2}(36)\frac{\pi}{3} = 6\pi \text{ cm}^2$.
8. $\binom{7}{2} = \frac{7 \times 6}{2} = 21$.

Next step

Heurisco's *ESAT Maths 2: Applied Reasoning and Modelling Practice Set* has twenty original questions in this style, with diagrams, from friendly to demanding, and a full worked solution for each.