

TMUA

Paper 1

Special Edition Solutions

Answer key, worked solutions and score guide.

Questions	20
Time allowed	75 minutes
Calculator	Not allowed
Marking	1 mark per question, no penalty

Answer key

1 E 2 E 3 B 4 E 5 C 6 A 7 E 8 D 9 C
 10 A 11 C 12 D 13 F 14 B 15 D 16 B 17 F 18 C
 19 F 20 B

Specification codes tested: 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8.

Score guide

This table is a rough planning aid. It is **not** an official score: UAT-UK does not publish a conversion for our papers, and real conversions change with every sitting.

Score	Percentage	Suggested next step
0–6	0–30%	Go back to the free lessons for this module, then try the topic sets.
7–10	35–50%	Fix the topics you missed with topic sets, then retake a different mock.
11–14	55–70%	Solid base. Work on speed and the questions you guessed.
15–17	75–85%	Strong. Practise full-length papers back to back.
18–20	90–100%	Excellent. Check for careless slips, then keep timing sharp.

Worked solutions

1. Answer E

1.1

Question: For how many integers k is $x^2 - 2kx + k + 2 \geq 0$ true for every real number $x \geq 0$?

The graph is a U-parabola with vertex at $x = k$; we only need $x \geq 0$, and the answer depends on where the vertex is. If $k \leq 0$ the vertex is not in the range and f is increasing for $x \geq 0$, so the least value is $f(0) = k + 2 \geq 0$: $-2 \leq k \leq 0$. If $k > 0$ the least value is at the vertex, $f(k) = -k^2 + k + 2 \geq 0$, i.e. $(k - 2)(k + 1) \leq 0$: $0 < k \leq 2$. Together $-2 \leq k \leq 2$: five integers.

Trap. 4 uses the condition 'discriminant ≤ 0 ' alone ($-1 \leq k \leq 2$), which is the condition for the inequality to hold for all real x and wrongly rejects $k = -2$; 3 and 7 drop or add endpoints.

2. Answer E

1.5

Question: For how many integers k does the equation $4^x - 2^{x+2} = k$ have exactly two distinct real solutions?

Let $u = 2^x > 0$: the equation becomes $u^2 - 4u = k$, and each positive u gives exactly one x . The graph of $y = u^2 - 4u = (u - 2)^2 - 4$, for $u > 0$, falls from 0 (not reached, since $u = 0$ is excluded) to -4 at $u = 2$ and then rises. A horizontal line $y = k$ meets it twice exactly when $-4 < k < 0$; at $k = -4$ it touches once, and at $k = 0$ the roots are $u = 0$ (rejected) and $u = 4$, only one solution. The integers $-3, -2, -1$ give 3.

Trap. 5 counts every k for which the quadratic in u has two real roots ($-4 < k \leq 0$ plus the touching case); 4 counts $k = 0$, forgetting that $u = 2^x$ can never be 0.

3. Answer B

1.4

Question: How many real numbers x with $0 \leq x \leq 2\pi$ satisfy $\tan^2 x = 3 \sin x$?

Write $\tan^2 x = \frac{\sin^2 x}{\cos^2 x}$ with $\cos x \neq 0$, and clear the denominator: $\sin^2 x = 3 \sin x \cos^2 x$, so $\sin x (\sin x - 3 \cos^2 x) = 0$. Do not divide by $\sin x$. (i) $\sin x = 0$: $x = 0, \pi, 2\pi$ (the interval is closed): 3 solutions. (ii) $\sin x = 3(1 - \sin^2 x)$, i.e. $3s^2 + s - 3 = 0$ with $s = \sin x$: $s = \frac{-1 \pm \sqrt{37}}{6}$. Only $s = \frac{-1 + \sqrt{37}}{6} \approx 0.85$ lies in $[-1, 1]$ (the other is below -1), and it gives two values of x in $(0, \pi)$. These have $\cos x \neq 0$. Total $3 + 2 = 5$.

Trap. 2 divides by $\sin x$ and loses the three solutions of $\sin x = 0$; 4 forgets one of $0, \pi, 2\pi$ (an endpoint); 1 and 3 come from also forgetting the second solution with $\sin x = \frac{-1 + \sqrt{37}}{6}$ in $(0, \pi)$.

4. Answer E

1.2

Question: Let $S_n = \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{n(n+2)}$. What is the smallest positive integer n for which $S_n > \frac{37}{50}$?

Split each term: $\frac{1}{k(k+2)} = \frac{1}{2} \left(\frac{1}{k} - \frac{1}{k+2} \right)$. The sum telescopes: $S_n = \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \right) = \frac{3}{4} - \frac{1}{2} \left(\frac{1}{n+1} + \frac{1}{n+2} \right)$. Since $\frac{3}{4} - \frac{37}{50} = \frac{1}{100}$, the condition is $\frac{1}{n+1} + \frac{1}{n+2} < \frac{1}{50}$. Test exactly: $n = 98$: $\frac{1}{99} + \frac{1}{100} = \frac{199}{9900} > \frac{198}{9900} = \frac{1}{50}$ fails; $n = 99$: $\frac{1}{100} + \frac{1}{101} = \frac{201}{10100} < \frac{202}{10100} = \frac{1}{50}$ works. The sum is increasing, so the smallest n is 99.

Trap. 100 comes from the estimate $\frac{2}{n+1.5} \approx \frac{1}{50}$ rounded the safe way; 98 is the value where the two sides are within $\frac{1}{9900}$ and fails by a hair; the whole question is decided by exact comparison.

5. Answer C

1.1

Question: For each real number a consider the equation $(x - a)(x^2 - 3x + a) = 0$. What is the sum of all the values of a for which the equation has exactly two distinct real solutions?

The solutions are $x = a$ together with the roots of $x^2 - 3x + a = 0$, whose discriminant is $9 - 4a$. If $a > \frac{9}{4}$ the quadratic has no real root: only one solution. If $a < \frac{9}{4}$ the quadratic has two distinct roots, so we get three solutions *unless* $x = a$ is one of them: $a^2 - 3a + a = 0$, i.e. $a = 0$ or $a = 2$ (both $< \frac{9}{4}$), which gives exactly two solutions. If $a = \frac{9}{4}$ the quadratic has the double root $\frac{3}{2} \neq a$: two solutions. So $a = 0, 2, \frac{9}{4}$ and the sum is $\frac{17}{4}$.

Trap. $2 = 0 + 2$ finds the coincidences but forgets the double-root case $a = \frac{9}{4}$; $\frac{9}{4}$ alone forgets the coincidences where the linear factor shares a root with the quadratic.

6. Answer A

1.8

Question: The function f is defined for all real x by $f(x) = x^3 + x^2 + x$. It is increasing, so it has an inverse f^{-1} . How many points do the graphs of $y = f(x)$ and $y = f^{-1}(x)$ have in common?

The two graphs are reflections of each other in $y = x$. Claim: for an increasing function, any common point lies on $y = x$. If (a, b) is on both graphs then $b = f(a)$ and $b = f^{-1}(a)$, i.e. $f(b) = a$. If $a < b$ then $f(a) < f(b)$ gives $b < a$, a contradiction; likewise $a > b$ is impossible, so $a = b$. So we need $f(x) = x$: $x^3 + x^2 = x^2(x + 1) = 0$, giving $x = 0$ and $x = -1$ (the double root at 0 is one point). Two points: $(0, 0)$ and $(-1, -1)$.

Trap. 3 counts the double root at $x = 0$ twice; 4 or more imagine extra pairs of points symmetric about $y = x$, which cannot occur for an increasing function; 1 misses $x = -1$.

7. Answer E

1.3

Question: ABC is an equilateral triangle with side 2. Three circular arcs are drawn, each bulging outwards from the triangle: one with centre A joining B to C , one with centre B joining C to A , and one with centre C joining A to B . They bound a rounded, triangle-like shape. What is the area of the region bounded by the three arcs?

Each arc has radius 2 and subtends 60° at its centre, so each sector (centre A, B or C) has area $\frac{1}{6}\pi \cdot 2^2 = \frac{2\pi}{3}$. The three sectors together cover the triangle ABC three times and each of the three curved caps once, so the region (triangle plus caps) has area (three sectors) $- 2 \times$ (triangle) $= 2\pi - 2 \cdot \sqrt{3}$, since the triangle has area $\frac{1}{2} \cdot 2 \cdot 2 \sin 60^\circ = \sqrt{3}$.

Trap. $2\pi - \sqrt{3}$ subtracts the triangle only once; $2\pi - 3\sqrt{3}$ is the three caps alone, without the triangle; 2π is the sum of the sectors.

8. Answer D

1.6

Question: P is the point $(1, 1)$ on the curve $y = x^3$. The tangent to the curve at P meets the curve again at Q , and the tangent at Q meets the curve again at R . What is the area of triangle PQR ?

The tangent at $x = p$ is $y = 3p^2(x - p) + p^3$. Where it meets $y = x^3$: $x^3 - 3p^2x + 2p^3 = 0$, which has the double root $x = p$ (tangency); the three roots add to 0 (no x^2 term), so the third root is $-2p$. So $x_Q = -2$ and then $x_R = -2 \cdot (-2) = 4$. The points are $P(1, 1)$, $Q(-2, -8)$, $R(4, 64)$. Area $= \frac{1}{2} |(-3)(63) - (-9)(3)| = \frac{1}{2} \cdot 162 = 81$.

Trap. 162 is the cross product without the factor $\frac{1}{2}$; $\frac{81}{2}$ halves twice; 54 and 27 come from taking $x_R = -2x_Q$ with a sign slip or $x_R = 2$.

9. Answer C

1.7

Question: The line $y = mx + 2$, where $m > 0$, meets the parabola $y = x^2$ at two points. The finite region between the line and the parabola has area 36. What is the value of m ?

Let the intersections have x -coordinates $\alpha < \beta$, roots of $x^2 - mx - 2 = 0$. The area between a line and a parabola $y = x^2$ is $\int_{\alpha}^{\beta} (mx + 2 - x^2) dx = \int_{\alpha}^{\beta} -(x - \alpha)(x - \beta) dx = \frac{(\beta - \alpha)^3}{6}$. (Memorise: width cubed over six.) Here $(\beta - \alpha)^2 = (\alpha + \beta)^2 - 4\alpha\beta = m^2 + 8$. So $\frac{(m^2 + 8)^{3/2}}{6} = 36$, $(m^2 + 8)^{3/2} = 216 = 6^3$, $m^2 + 8 = 36$, $m^2 = 28$, $m = 2\sqrt{7}$.

Trap. $4\sqrt{2}$ uses $(\beta - \alpha)^2 = m^2 + 4$ (a slip in the discriminant: $-4\alpha\beta = +8$, not $+4$); the others are near misses.

10. Answer A

1.7

Question: The trapezium rule with n strips of equal width is used to estimate $\int_0^2 (x^3 - 3x^2) dx$. For which positive integers n is the estimate exactly equal to the true value of the integral?

Put $x = 1 + t$, the midpoint of the interval: $x^3 - 3x^2 = (1 + t)^3 - 3(1 + t)^2 = t^3 - 3t - 2$. So the graph is the odd function $t^3 - 3t$ shifted down by 2: it is symmetric under a half-turn about $(1, -2)$, the middle of the interval. The integral is $-2 \times 2 = -4$, since the odd part cancels. For the trapezium rule, the nodes are symmetric about $x = 1$ for every n , and the rule's weights are symmetric too, so the function values at mirror-image nodes always average to -2 ; the estimate is again $-2 \times 2 = -4$. (For $n = 1$ the chord from $(0, 0)$ to $(2, -4)$ passes through the centre $(1, -2)$ of symmetry; the same is true for $n = 2, 3, \dots$) Exact for every n .

Trap. 'n = 1 only', 'n = 1, 2 only' and 'even n only' come from testing a few values of n (or from needing a node at the centre), instead of using the half-turn symmetry; 'no value of n ' comes from the expectation that the rule over- or under-estimates a curved graph.

11. Answer C

1.8

Question: Consider $y = |x^2 - 4x| - 3$. For each real number k , let $N(k)$ be the number of real solutions of $||x^2 - 4x| - 3| = k$. What is the value of $N(0) + N(1) + N(2) + N(3) + N(4)$?

Track $p = |x^2 - 4x|$: it is $4x - x^2$ on $[0, 4]$ (a hump of height 4 at $x = 2$) and $x^2 - 4x$ outside. The graph of $y = |p - 3|$ has four zeros, where $p = 3$: $x = 2 \pm \sqrt{7}$ and $x = 1, 3$. Reading it from the left: it comes down to 0 at $x = 2 - \sqrt{7}$, rises to a peak of height 3 at $x = 0$, falls to 0 at $x = 1$, rises to a small peak of height $|4 - 3| = 1$ at $x = 2$, falls to 0 at $x = 3$, rises to a peak of height 3 at $x = 4$, falls to 0 at $x = 2 + \sqrt{7}$ and then rises for ever. Slide a horizontal line: $k = 0$: 4 solutions (the zeros); $0 < k < 1$: 8; $k = 1$: the line touches the small peak once instead of cutting it twice, giving 7; $1 < k < 3$ (e.g. $k = 2$): 6; $k = 3$: the line touches the two outer peaks at $x = 0, 4$ and cuts each outer arm once: 4; $k = 4$: only the two outer arms: 2. Total $4 + 7 + 6 + 4 + 2 = 23$.

Trap. 24 counts 8 solutions at $k = 1$ (not noticing the touching at $x = 2$ gives 7); 22 drops a solution at $k = 3$ or $k = 0$ by missing an endpoint of a hump.

12. Answer D

1.3

Question: The points $A(-2, 0)$ and $B(2, 0)$ are fixed. For each real number c , consider the points P on the line $y = x + c$ for which angle APB is exactly 90° . For how many values of c is there exactly one such point P ?

$\angle APB = 90^\circ$ exactly when P lies on the circle with diameter AB , i.e. $x^2 + y^2 = 4$, but $P \neq A, B$ (at those points there is no angle). The line $y = x + c$ meets the circle in two points when $|c| < 2\sqrt{2}$, once (tangent) when $c = \pm 2\sqrt{2}$, and never beyond. For $c = \pm 2\sqrt{2}$: one point, valid. For $|c| < 2\sqrt{2}$ there are two intersection points, but if one of them is A or B it must be discarded, leaving one: the line passes through $A(-2, 0)$ when $c = 2$ and through $B(2, 0)$ when $c = -2$. So the values are $c = 2\sqrt{2}, -2\sqrt{2}, 2, -2$: four.

Trap. 2 finds only the two tangent lines and forgets that the lines through A or B also leave exactly one valid point; 3 remembers only one of A, B .

13. Answer F

1.2

Question: How many real numbers x with $0 \leq x < 2\pi$ satisfy $\cot x + \cot^2 x + \cot^3 x + \dots = 2 \tan x$, where the left-hand side is the sum of the infinite series?

Let $c = \cot x$. The series is geometric with ratio c , so it converges (to $\frac{c}{1-c}$) only if $|c| < 1$. The equation becomes $\frac{c}{1-c} = \frac{2}{c}$ (as $\tan x = \frac{1}{c}$), so $c^2 = 2 - 2c$, i.e. $c^2 + 2c - 2 = 0$ and $c = -1 \pm \sqrt{3}$. Only $c = \sqrt{3} - 1 \approx 0.73$ satisfies $|c| < 1$; the other root, $-1 - \sqrt{3}$, makes the series diverge. Now $\cot x = \sqrt{3} - 1 > 0$ has one solution in each of the intervals $(0, \frac{\pi}{2})$ and $(\pi, \frac{3\pi}{2})$: two solutions.

Trap. 4 accepts both roots of the quadratic (two values of x each) without testing the convergence condition $|\cot x| < 1$.

14. Answer B

1.2

Question: A sequence is defined by $u_1 = 3$ and $u_{n+1} = u_n^2 - 2u_n + 2$ for $n \geq 1$. What is the units digit of u_{2025} ?

Notice $u_{n+1} - 1 = u_n^2 - 2u_n + 1 = (u_n - 1)^2$. So $v_n = u_n - 1$ satisfies $v_{n+1} = v_n^2$ with $v_1 = 2$, giving $v_n = 2^{2^{n-1}}$ and $u_n = 2^{2^{n-1}} + 1$ (the terms are 3, 5, 17, 257, 65537, ...). For $n = 2025$ the exponent 2^{2024} is a multiple of 4, and $2^{4j} = 16^j$ ends in 6. So u_{2025} ends in $6 + 1 = 7$.

Trap. 3 and 5 are the units digits of the first two terms; 6 forgets the $+1$; 1 and 9 come from mis-cycling the units digits of powers of 2 (period 4) with an exponent that is itself a power of 2.

15. Answer D

1.3

Question: The point $P(3, 1)$ lies inside the circle $x^2 + y^2 = 25$. A *chord* of the circle is a line segment joining two points of the circle. How many chords of the circle that pass through P have an integer length?

A chord through P at distance d from the centre O has length $2\sqrt{25 - d^2}$. As the chord turns about P , d runs from 0 (the diameter through P , length 10) up to $OP = \sqrt{10}$ (the chord perpendicular to OP , the shortest, length $2\sqrt{25 - 10} = 2\sqrt{15} = \sqrt{60}$). Since $7^2 < 60 < 8^2$, the lengths available are all values in $(7.7 \dots, 10]$. The integers are 8, 9, 10. A length strictly between the extremes occurs for exactly two chords (mirror images in the line OP), but the maximum 10 occurs only once. So the count is $2 + 2 + 1 = 5$.

Trap. 3 counts the integer lengths 8, 9, 10 rather than the chords; 4 forgets the diameter; 6 counts the diameter twice.

16. Answer B

1.2

Question: Two perpendicular lines meet at a point O . The circle C_0 of radius 1 touches both lines. The circle C_1 touches both lines and touches C_0 , and lies between C_0 and O . The circle C_2 touches both lines and C_1 , and lies between C_1 and O , and so on for ever. What is the total area of all the circles $C_0, C_1, C_2, \dots$?

By symmetry all centres lie on the line $y = x$ (taking the lines as axes): a circle of radius r has centre (r, r) . For consecutive radii $r > s$ the circles touch externally, so $\sqrt{2}(r - s) = r + s$, giving $s = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} r = (3 - 2\sqrt{2})r$. So the radii form a geometric sequence with ratio $q = 3 - 2\sqrt{2}$, and the areas πr_n^2 form a geometric series with first term π and ratio $q^2 = 17 - 12\sqrt{2}$. The sum is $\frac{\pi}{1 - (17 - 12\sqrt{2})} = \frac{\pi}{12\sqrt{2} - 16} = \frac{\pi(3\sqrt{2} + 4)}{8}$.

Trap. $\frac{\pi(1+\sqrt{2})}{2}$ uses the ratio q for the *areas* instead of q^2 ; $\pi(1 + \sqrt{2})$ is the total *circumference* of all the circles divided by 2 times the radius sum, i.e. the sum of πr_n rather than πr_n^2 ; $\frac{9\pi}{8}$ and $\frac{4\pi}{3}$ come from guessing ratios $\frac{1}{3}, \frac{1}{2}$.

17. Answer F

1.7

Question: For a constant $q > 1$, the curve $y = x(x-1)(x-q)$ and the x -axis enclose two finite regions. The two regions have equal area. What is the area of one of them?

Equal areas mean the signed integral from 0 to q is zero. By symmetry that happens when the curve is odd about its middle root $x = 1$: put $x = 1 + t$, so $y = (1+t)t(1+t-q)$; for $q = 2$ this is $t(t+1)(t-1) = t^3 - t$, an odd function, so the two lobes are congruent. (Integrating confirms it: $\int_0^q y dx = \frac{q^3(2-q)}{12}$, zero only for $q = 2$.) So $q = 2$ and the area of one lobe is $\int_0^1 (x^3 - 3x^2 + 2x) dx = \frac{1}{4} - 1 + 1 = \frac{1}{4}$.

Trap. $\frac{1}{2}$ is the total area of both lobes; $\frac{5}{12}$ is the lobe $\int_0^1$ for the wrong guess $q = 3$ (picking q without checking equal areas); 1 and $\frac{3}{4}$ are unjustified guesses.

18. Answer C

1.7

Question: The function f is defined by $f(x) = x^3 + x$ for all real x . It is increasing, so it has an inverse function f^{-1} . What is the value of $\int_0^2 f^{-1}(x) dx$?

Do not find f^{-1} . The graph of $y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$, so the area under $y = f^{-1}(x)$ for $0 \leq x \leq 2$ equals the area between the curve $y = f(x)$ and the y -axis for $0 \leq y \leq 2$. Now $f(0) = 0$ and $f(1) = 2$, so this region fills the rectangle $0 \leq x \leq 1, 0 \leq y \leq 2$ (area 2) except for the part under the curve, which has area $\int_0^1 (x^3 + x) dx = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$. The required integral is $2 - \frac{3}{4} = \frac{5}{4}$.

Trap. $\frac{3}{4}$ is $\int_0^1 f$, the area on the wrong side of the curve; $\frac{7}{4} = 2 - \frac{1}{4}$ forgets the x term when integrating f .

19. Answer F

1.8

Question: What is the area of the region of the plane defined by $|x| + |y| + |x - y| \leq 4$?

Look at the three numbers $x, -y, y - x$: they add to 0. When three real numbers sum to 0, two of them share a sign and the third is minus their sum, so the one of largest size has modulus equal to the sum of the other two moduli. Hence $|x| + |y| + |x - y| = 2 \max(|x|, |y|, |x - y|)$, and the condition is $|x| \leq 2, |y| \leq 2, |x - y| \leq 2$. That is the square $-2 \leq x, y \leq 2$ (area 16) with two opposite corners cut off by $|x - y| \leq 2$: the corner triangles near $(2, -2)$ and $(-2, 2)$ each have legs 2 (area 2). The region is a hexagon with vertices $(2, 0), (2, 2), (0, 2), (-2, 0), (-2, -2), (0, -2)$, and area $16 - 2 \cdot 2 = 12$.

Trap. 16 keeps only the square $|x| \leq 2, |y| \leq 2$ and forgets $|x - y| \leq 2$ altogether; 14 cuts off only one of the two corner triangles; 32 is the area of the diamond $|x| + |y| \leq 4$ (the term $|x - y|$ ignored).

20. Answer B

1.6

Question: The curve C has equation $y = x^3 - 3x$. For certain real numbers k , exactly two different tangent lines to C pass through the point $(2, k)$. What is the sum of all such values of k ?

The tangent at the point with x -coordinate t is $y = (3t^2 - 3)(x - t) + t^3 - 3t$. It passes through $(2, k)$ exactly when $k = (3t^2 - 3)(2 - t) + t^3 - 3t = g(t)$, where $g(t) = -2t^3 + 6t^2 - 6$. So the number of tangents through $(2, k)$ is the number of solutions t of $g(t) = k$: sketch $y = g(t)$ and slide a horizontal line. Now $g'(t) = -6t(t - 2)$, so g has a local minimum $g(0) = -6$ and a local maximum $g(2) = 2$. A horizontal line meets this cubic in three points for $-6 < k < 2$, in one point for $k < -6$ or $k > 2$, and in exactly two points when it touches a turning point, $k = -6$ or $k = 2$. The sum is $-6 + 2 = -4$.

Trap. -6 and 2 each give only one of the two special values (a local minimum or a local maximum of g); -2 is the value $g(2)$ with a sign slip, and 0 is the case $(2, 0)$, which has three tangents.