

TMUA

Paper 2

Free Mock Paper Solutions

Answer key, worked solutions and score guide.

Questions	20
Time allowed	75 minutes
Calculator	Not allowed
Marking	1 mark per question, no penalty

Answer key

1 A 2 F 3 B 4 C 5 E 6 D 7 A 8 C 9 G
 10 F 11 F 12 A 13 B 14 E 15 B 16 E 17 C 18 H
 19 D 20 D

Specification codes tested: 1.1, 1.2, 1.3, 1.4, 1.5, 1.6, 1.7, 1.8, 2.1, 2.2, 2.3, 2.4, 2.5.

Score guide

This table is a rough planning aid. It is **not** an official score: UAT-UK does not publish a conversion for our papers, and real conversions change with every sitting.

Score	Percentage	Suggested next step
0–6	0–30%	Go back to the free lessons for this module, then try the topic sets.
7–10	35–50%	Fix the topics you missed with topic sets, then retake a different mock.
11–14	55–70%	Solid base. Work on speed and the questions you guessed.
15–17	75–85%	Strong. Practise full-length papers back to back.
18–20	90–100%	Excellent. Check for careless slips, then keep timing sharp.

Worked solutions

1. Answer A

1.7

Question: The curve $y = x(6 - x)$ and the line $y = 2x$ enclose a finite region. What is its area?

The curve and line meet where $6x - x^2 = 2x$, that is $x(4 - x) = 0$, so $x = 0$ and $x = 4$. Between these values the curve lies above the line (at $x = 2$ the curve is at 8 and the line at 4).

$$\text{Area} = \int_0^4 (6x - x^2 - 2x) dx = \int_0^4 (4x - x^2) dx = \left[2x^2 - \frac{x^3}{3} \right]_0^4 = 32 - \frac{64}{3} = \frac{32}{3}.$$

Trap. $\frac{80}{3}$: integrating only $6x - x^2$ from 0 to 4, which is the area under the curve, and forgetting to subtract the area under the line.

2. Answer F

1.1

Question: The number $\frac{\sqrt{7} + \sqrt{3}}{\sqrt{7} - \sqrt{3}}$ can be written as $p + q\sqrt{21}$, where p and q are rational numbers. What is the value of $p + q$?

Multiply top and bottom by $\sqrt{7} + \sqrt{3}$:

$$\frac{(\sqrt{7} + \sqrt{3})^2}{(\sqrt{7})^2 - (\sqrt{3})^2} = \frac{7 + 2\sqrt{21} + 3}{4} = \frac{10 + 2\sqrt{21}}{4} = \frac{5}{2} + \frac{1}{2}\sqrt{21}.$$

So $p = \frac{5}{2}$, $q = \frac{1}{2}$ and $p + q = 3$.

Trap. 6: dividing by 2 instead of 4 (the denominator is $7 - 3 = 4$), which gives $5 + \sqrt{21}$.

3. Answer B

1.4

Question: What is the sum of all the solutions, in degrees, of the equation $3 \sin x = 2 \cos^2 x$ in the interval $0 \leq x \leq 720$?

Using $\cos^2 x = 1 - \sin^2 x$, the equation becomes $3 \sin x = 2 - 2 \sin^2 x$, that is $2 \sin^2 x + 3 \sin x - 2 = 0$, which factorises as $(2 \sin x - 1)(\sin x + 2) = 0$.

Since $\sin x \neq -2$, we need $\sin x = \frac{1}{2}$. In one turn the solutions are $x = 30$ and $x = 150$; in the second turn they are $x = 390$ and $x = 510$.

The sum is $30 + 150 + 390 + 510 = 1080$.

Trap. 180: finding only the solutions 30 and 150 in the first turn and not continuing to 720.

4. Answer C

1.2

Question: A geometric series has first term 12 and sum to infinity 16. What is the sum of the first three terms?

The sum to infinity is $\frac{a}{1-r} = \frac{12}{1-r} = 16$, so $1-r = \frac{3}{4}$ and $r = \frac{1}{4}$.

The first three terms are 12, 3, $\frac{3}{4}$, so the sum is $12 + 3 + \frac{3}{4} = \frac{63}{4}$.

Trap. $\frac{255}{16}$: using the sum formula $\frac{a(1-r^n)}{1-r}$ with $n = 4$, an off-by-one slip. (15 is the sum of just the first two terms.)

5. Answer E

1.3

Question: The circle C has equation $x^2 + y^2 - 4x + 6y = 12$, and the point $(6, 0)$ lies on C . The tangent to C at $(6, 0)$ and the two coordinate axes enclose a triangle. What is the area of this triangle?

Completing the square: $(x - 2)^2 + (y + 3)^2 = 25$, so the centre is $(2, -3)$. The radius to $(6, 0)$ has gradient $\frac{0 - (-3)}{6 - 2} = \frac{3}{4}$.

The tangent is perpendicular to the radius, so its gradient is $-\frac{4}{3}$, and its equation is $y = -\frac{4}{3}(x - 6)$, that is $4x + 3y = 24$.

This meets the axes at $(6, 0)$ and $(0, 8)$, so together with the origin it bounds a right-angled triangle of area $\frac{1}{2} \times 6 \times 8 = 24$.

Trap. $\frac{27}{2}$: taking the tangent gradient as $-\frac{3}{4}$ (the negative of the radius gradient rather than the negative reciprocal), giving $3x + 4y = 18$ and a triangle with legs 6 and $\frac{9}{2}$.

6. Answer D

1.6

Question: For each integer k , the curve $y = x^3 + kx^2 + 12x + 1$ is drawn. For how many integer values of k does the curve have no stationary points?

The gradient is $\frac{dy}{dx} = 3x^2 + 2kx + 12$. There are no stationary points exactly when this quadratic has no real roots, that is when its discriminant is negative: $4k^2 - 144 < 0$, so $k^2 < 36$ and $-6 < k < 6$.

The integers are $-5, -4, \dots, 5$: 11 values.

Trap. 13: using $k^2 \leq 36$. For $k = \pm 6$ the gradient is $3(x \pm 2)^2$, which is zero at $x = \mp 2$, so there is a stationary point (a point of inflection).

7. Answer A

1.5

Question: The positive real numbers x and y satisfy $\log_4 x + \log_2 y = 4$ and $\log_2 x - \log_4 y = 3$. What is the value of $\frac{x}{y}$?

Let $X = \log_2 x$ and $Y = \log_2 y$. Since $\log_4 x = \frac{\log_2 x}{\log_2 4} = \frac{1}{2}X$, the equations become

$$\frac{1}{2}X + Y = 4, \quad X - \frac{1}{2}Y = 3.$$

From the second, $X = 3 + \frac{1}{2}Y$. Substituting, $\frac{3}{2} + \frac{1}{4}Y + Y = 4$, so $\frac{5}{4}Y = \frac{5}{2}$ and $Y = 2$, then $X = 4$.

Hence $x = 2^4 = 16$ and $y = 2^2 = 4$, so $\frac{x}{y} = 4$.

Trap. 16: stopping at the value of x (or of y^2) without dividing by y ; (12 is $x - y$.)

8. Answer C

2.1

Question: For a positive integer n , let P be the statement “ $n^2 - 1$ is a multiple of 8” and let Q be the statement “ n is odd”. Which of the following is true?

Sufficient. If n is odd, then $n - 1$ and $n + 1$ are consecutive even numbers, so one of them is a multiple of 4 and the other is a multiple of 2. Hence $n^2 - 1 = (n - 1)(n + 1)$ is a multiple of 8.

Necessary. If n is even, then $n^2 - 1$ is odd, so it is not a multiple of 8. Hence if $n^2 - 1$ is a multiple of 8, n must be odd.

So Q is both necessary and sufficient for P .

Trap. “necessary but not sufficient”: thinking that being odd is not enough to make $n^2 - 1$ a multiple of 8 (for instance that $n - 1$ itself would have to be a multiple of 4). The two factors $n - 1$ and $n + 1$ between them always supply the extra factor.

9. Answer G

1.8

Question: The function f is defined by $f(x) = x^2 - 6x + 5$ for all real x . Which of the following statements are true?

- I The graph of $y = f(x - 2)$ crosses the x -axis at $x = -1$ and $x = 3$.
- II The graph of $y = f(2x)$ crosses the x -axis at $x = \frac{1}{2}$ and $x = \frac{5}{2}$.
- III The graph of $y = f(-x)$ crosses the x -axis at $x = -5$ and $x = -1$.

Since $f(x) = (x - 1)(x - 5)$, $f(u) = 0$ exactly when $u = 1$ or $u = 5$.

I. $f(x - 2) = 0$ when $x - 2 = 1$ or $x - 2 = 5$, that is $x = 3$ or $x = 7$. (The graph is translated 2 units to the right.) False.

II. $f(2x) = 0$ when $2x = 1$ or $2x = 5$, that is $x = \frac{1}{2}$ or $x = \frac{5}{2}$. True.

III. $f(-x) = 0$ when $-x = 1$ or $-x = 5$, that is $x = -1$ or $x = -5$. True.

So II and III only.

Trap. "I, II and III": reading $f(x - 2)$ as a shift of 2 units to the left, which would give the crossing points -1 and 3 .

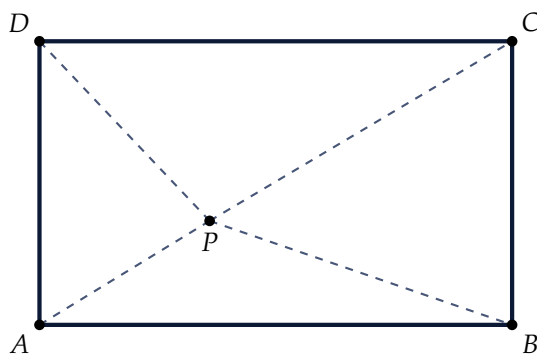
10. Answer F

2.3

Question: The diagram shows a rectangle $ABCD$ and a point P inside it. The rectangle and the point shown are not special in any way. Consider the following statements, which refer to any rectangle $ABCD$ and any point P inside it.

- I $PA^2 + PC^2 = PB^2 + PD^2$.
- II If $PA = PC$, then P lies on the diagonal BD .
- III The areas of the triangles PAB and PCD add up to half the area of the rectangle.

Which of these statements are true?



Take axes with $A = (0, 0)$, $B = (w, 0)$, $C = (w, h)$, $D = (0, h)$ and $P = (p, q)$.

I is true. $PA^2 + PC^2 = p^2 + q^2 + (w - p)^2 + (h - q)^2$ and $PB^2 + PD^2 = (w - p)^2 + q^2 + p^2 + (h - q)^2$: the same.

II is false. $PA = PC$ means that P lies on the perpendicular bisector of AC . This passes through the centre of the rectangle but is perpendicular to AC , whereas the diagonal BD is perpendicular to AC only when the rectangle is a square. For $w = 5$, $h = 3$ the bisector is the line $5p + 3q = 17$ and BD is the line $3p + 5q = 15$. The point $P = (2, \frac{7}{3})$ lies inside the rectangle and on the first line, but $3 \times 2 + 5 \times \frac{7}{3} \neq 15$, so it is not on BD although $PA = PC$.

III is true. The triangles PAB and PCD have bases $AB = CD = w$ and heights q and $h - q$, so together their areas are $\frac{1}{2}wq + \frac{1}{2}w(h - q) = \frac{1}{2}wh$.

So I and III only.

Trap. “I, II and III”: noting that the perpendicular bisector of AC goes through the centre of the rectangle, as BD does, and concluding that they are the same line. They coincide only for a square.

11. Answer F

2.3

Question: A student claims: “if a and b are positive numbers with $a > b$, then $a^b > b^a$ ”. Which of the following pairs (a, b) shows that the claim is false?

A counterexample must have $a > b > 0$ but $a^b \leq b^a$.

For $(4, 3)$: $a^b = 4^3 = 64$ and $b^a = 3^4 = 81$, and $64 < 81$, so the claim fails.

The other pairs satisfy the claim: $2^1 = 2 > 1^2 = 1$; $3^2 = 9 > 2^3 = 8$; $5^1 = 5 > 1^5 = 1$; $3^{1/2} = \sqrt{3} > \left(\frac{1}{2}\right)^3 = \frac{1}{8}$; and $9^{1/3} > 1 > \left(\frac{1}{3}\right)^9$ (a number greater than 1 against a number less than 1).

Trap. $(9, \frac{1}{3})$: thinking a fractional exponent makes $9^{1/3}$ tiny. In fact $9^{1/3} \approx 2.1$ is greater than 1, whereas $\left(\frac{1}{3}\right)^9$ is far below 1.

12. Answer A

2.4

Question: A student solves the inequality $\frac{x+1}{x-2} > 3$ for real x as follows.

- (1) Multiplying both sides by $x - 2$ gives $x + 1 > 3(x - 2)$.
- (2) Expanding, $x + 1 > 3x - 6$.
- (3) Rearranging, $7 > 2x$.
- (4) Hence $x < \frac{7}{2}$.
- (5) The solutions are therefore all real numbers $x < \frac{7}{2}$ with $x \neq 2$.

Which of the following is correct?

Line (1) is wrong. Multiplying an inequality by $x - 2$ preserves the direction only if $x - 2 > 0$; if $x - 2 < 0$ the inequality sign must be reversed. The student has not split into the two cases.

Lines (2), (3) and (4) are correct algebra. Line (5) is wrong, but only as a consequence of line (1): for example $x = 0$ gives $\frac{1}{-2} = -\frac{1}{2}$, which is not greater than 3.

(The correct solution is $\frac{7-2x}{x-2} > 0$, that is $2 < x < \frac{7}{2}$.)

Trap. “The first error is in line (5)”: noticing that the final answer contains values such as $x = 0$ that do not work, and blaming the last line rather than the multiplication in line (1) that caused it.

13. Answer B

2.2

Question: In the following statements, m, n and k are integers and x is a real number. Which of the statements are true?

- I For every real number x there is an integer n such that $n > x$.
- II There is an integer n such that $n > x$ for every real number x .
- III For all integers m and n with $m < n$ there is an integer k such that $m < k < n$.

I is true: given x , take for example $n = \lfloor x \rfloor + 1$.

II is false: whichever integer n is chosen, the real number $x = n$ (or any $x \geq n$) has $n > x$ false.

III is false: take $m = 1$, $n = 2$. There is no integer strictly between 1 and 2.

So I only. (Statements I and II differ only in the order of “for every” and “there is”, and the order matters.)

Trap. “I and II only”: reading II as a repeat of I. In I the integer n may depend on x ; in II a single n would have to beat every real number.

14. Answer E

2.5

Question: A two-digit number N has tens digit a and units digit b , where a and b are non-zero. The number N' is formed by reversing the digits of N , so $N = 10a + b$ and $N' = 10b + a$. For example, if $N = 42$ then $N' = 24$. Which of the following must be true?

I $N + N'$ is a multiple of 11.

II $N - N'$ is a multiple of 9.

III $N + N'$ is a multiple of 9.

I. $N + N' = 11a + 11b = 11(a + b)$. True.

II. $N - N' = 9a - 9b = 9(a - b)$. True.

III. $N + N' = 11(a + b)$ is a multiple of 9 only when $a + b$ is a multiple of 9. For $N = 12$, $N + N' = 33$, which is not. False.

So I and II only.

Trap. “I, II and III”: extending the pattern of II to the sum. The difference carries a factor 9, the sum carries a factor 11.

15. Answer B

2.1

Question: For real numbers b and c , let P be the statement “the equation $x^2 + bx + c = 0$ has two distinct positive real roots” and let Q be the statement “ $b < 0$ and $c > 0$ ”. Which of the following is true?

Necessary. Suppose the roots are $\alpha, \beta > 0$. Then $x^2 + bx + c = (x - \alpha)(x - \beta)$, so $b = -(\alpha + \beta) < 0$ and $c = \alpha\beta > 0$. So P implies Q .

Not sufficient. Take $b = -1$ and $c = 1$. Then $b < 0$ and $c > 0$, but the discriminant is $b^2 - 4c = -3 < 0$, so there are no real roots at all.

So Q is necessary but not sufficient.

Trap. “both necessary and sufficient”: remembering that for a quadratic the sum of the roots is $-b$ and the product is c , but forgetting that the roots must also be real and distinct, which needs $b^2 > 4c$.

16. Answer E

2.3

Question: Three students each try to prove that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all real numbers $a \geq 0$ and $b \geq 0$.

I Both sides are non-negative, and $(\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab} \geq a + b = (\sqrt{a+b})^2$. Taking positive square roots of both ends gives the result.

II Suppose that $\sqrt{a+b} > \sqrt{a} + \sqrt{b}$. Squaring both sides gives $a + b > a + b + 2\sqrt{ab}$, so $\sqrt{ab} < 0$, which is impossible. So the statement is true.

III If $a = 1$, $b = 1$ then $\sqrt{2} < 2$; if $a = 4$, $b = 9$ then $\sqrt{13} < 5$; and if $a = 0$ the two sides are equal. So the statement holds for all a and b .

Which of these arguments is a valid proof?

I is valid: it starts from the true fact $2\sqrt{ab} \geq 0$ and, because both $\sqrt{a+b}$ and $\sqrt{a} + \sqrt{b}$ are non-negative, comparing squares is the same as comparing the numbers themselves.

II is valid: a proof by contradiction. If $X > Y \geq 0$ then $X^2 > Y^2$, so squaring the assumed inequality is legitimate, and it leads to the impossible $\sqrt{ab} < 0$.

III is not a proof: checking a few cases cannot establish a statement about all a and b .

So I and II only.

Trap. “I, II and III”: treating several confirming examples as a proof. A finite number of cases never proves a statement about all real a and b .

17. Answer C

2.5

Question: A positive integer is called *odd-handed* if it has an odd number of positive divisors (counting 1 and the number itself). For example, 9 is odd-handed since its divisors are 1, 3 and 9. How many positive integers from 1 to 100 inclusive are odd-handed?

Pair each divisor d of n with the divisor n/d . The pairs consist of two different numbers unless $d = n/d$, that is $d^2 = n$. So the number of divisors is odd exactly when there is such an unpaired divisor, which happens exactly when n is a perfect square.

The perfect squares from 1 to 100 are $1^2, 2^2, \dots, 10^2$: there are 10 of them.

Trap. 9: leaving out 1 (which has the single divisor 1 and is odd-handed) or leaving out 100. (25 is the number of primes up to 100, which have exactly two divisors and so are not odd-handed.)

18. Answer H

2.3

Question: The positive real numbers a and b satisfy $a + b = 1$. Which of the following must be true?

I $a^2 + b^2 \geq \frac{1}{2}$.

II $ab \leq \frac{1}{4}$.

III $a^3 + b^3 \geq \frac{1}{4}$.

II. $(a - b)^2 \geq 0$ gives $a^2 + b^2 \geq 2ab$, so $(a + b)^2 = a^2 + b^2 + 2ab \geq 4ab$, that is $4ab \leq 1$ and $ab \leq \frac{1}{4}$. True.

I. $a^2 + b^2 = (a + b)^2 - 2ab = 1 - 2ab \geq 1 - \frac{1}{2} = \frac{1}{2}$ by II. True.

III. $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 1 - 3ab \geq 1 - \frac{3}{4} = \frac{1}{4}$ by II. True.

(Equality holds in all three when $a = b = \frac{1}{2}$.) So I, II and III.

Trap. “I and II only”: doubting III because $a^3 + b^3$ looks as if it should be smaller than $a^2 + b^2$. It is, but its least value is still $\frac{1}{4}$, reached at $a = b = \frac{1}{2}$.

19. Answer D

2.5

Question: A counter moves along the lines of a grid from $(0, 0)$ to $(5, 5)$, each move being one unit to the right or one unit up. How many different routes avoid both of the points $(2, 2)$ and $(3, 3)$?

Without restriction there are $\binom{10}{5} = 252$ routes.

Routes through $(2, 2)$: $\binom{4}{2} \times \binom{6}{3} = 6 \times 20 = 120$. Routes through $(3, 3)$: $\binom{6}{3} \times \binom{4}{2} = 20 \times 6 = 120$. Routes through both: $6 \times 2 \times 6 = 72$ (from $(2, 2)$ to $(3, 3)$ there are 2 routes).

By inclusion–exclusion, $120 + 120 - 72 = 168$ routes touch at least one of the two points, so $252 - 168 = 84$ avoid both.

Trap. 12: subtracting 120 and 120 from 252 without adding back the 72 routes counted twice. (168 is the number of routes that do touch one of the points.)

20. Answer D

2.5

Question: Each of the nine cells of a 3×3 grid is filled with either 0 or 1. Each of the four 2×2 blocks of neighbouring cells contains exactly two 1s. How many different fillings of the grid are possible?

Case 1: every row alternates (each row is 010 or 101). Any two rows then give exactly one 1 in every pair of adjacent columns, so every 2×2 block has two 1s. There are $2^3 = 8$ such grids.

Case 2: some row has two equal neighbours, say a, a in columns $j, j+1$. A block containing them and the neighbouring row needs a total of 2, so the neighbouring row has $1-a, 1-a$ in those columns, and so on down the grid. So columns j and $j+1$ each alternate (e.g. 0, 1, 0 or 1, 0, 1). If a column alternates, so does the next: for entries a, x in a row and $1-a, y$ below, the block sum $1+x+y=2$ forces $y=1-x$. So every column alternates, which gives $2^3 = 8$ grids.

The two cases overlap in the grids where every row and every column alternates: the two chessboard patterns. By inclusion–exclusion the total is $8+8-2=14$.

Trap. 16: adding the 8 grids with alternating rows to the 8 with alternating columns without noticing that the two chessboard patterns have been counted twice.