

TMUA

Paper 1

Algebra and functions

Free lesson notes

Covers	1.1 algebra, polynomials, inequalities, modulus, surds, indices, functions
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

More than half of Paper 1 is algebra wearing a disguise. A question about a circle, a sequence or a tangent will, three lines in, turn into a quadratic, an inequality or a factor to find. If your algebra is quick and tidy you gain minutes; if it is slow you lose the whole paper.

The TMUA never asks you to learn a new method. It asks you to use GCSE and AS-level methods *cleverly*, in a form you have not seen before, and without a calculator. This lesson gives you (1) the small set of ideas that carry most of the marks, (2) a named strategy for each, and (3) ten worked examples built around the patterns that real papers keep returning to. Everything here is within the specification, and every answer was checked by computer as well as by hand.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

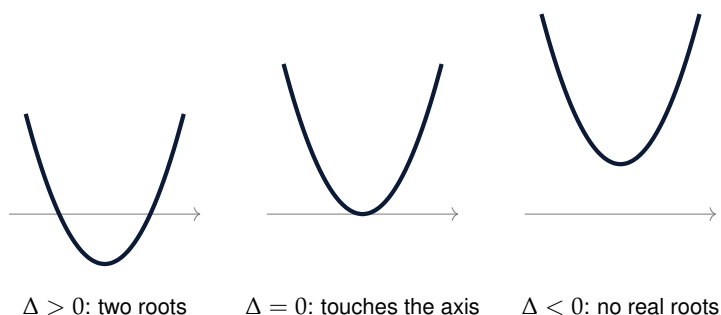
What the TMUA keeps asking

- **Discriminant questions** (almost every paper has two or three): “find the values of k for which...” a quadratic is always positive, a line touches or misses a curve, or a pair of simultaneous equations has two distinct solutions.
- **Quadratics in disguise**: a minimum or maximum of something like $4^x - 2^{x+3} + 4$ or $\frac{1}{5^{2x} - 4 \cdot 5^x + 7}$, solved by substitution and completing the square.
- **Roots and coefficients**: using sum and product of roots, or finding a new quadratic whose roots are built from the old ones (for example the square roots of the roots).
- **Factor and remainder theorem** with two unknowns, or a remainder that depends on the order in which 1, 2, 3 are assigned to coefficients.
- **Combined inequalities and the modulus**: two conditions that must hold together, or the greatest possible value of $|xy|$ given bounds on x and y .
- **Functions** defined by an unusual rule (“ $f(mn) = \dots$ if mn is...”), composition, inverses, and sequences of transformations applied in a stated order.

Notice what is *not* on this list: long expansions, heavy factorising, or any arithmetic that needs a calculator. If your working is getting ugly, you have missed the trick. Stop and look for it.

Quadratics and the discriminant

A quadratic $ax^2 + bx + c$ ($a \neq 0$) has the **discriminant** $\Delta = b^2 - 4ac$. It counts the real roots of $ax^2 + bx + c = 0$:



The picture above has $a > 0$. If $a < 0$ the parabola is upside down, but the three cases for the roots are the same. Three situations turn up again and again, and all three are the same idea.

1. The expression is positive (or negative) for all x . For $a > 0$, " $ax^2 + bx + c > 0$ for all real x " means the graph never reaches the axis, so $\Delta < 0$. For $a < 0$ and " < 0 for all x " it is also $\Delta < 0$.

2. A line and a curve. To find where $y = mx + c$ meets a curve, set them equal. You get a quadratic in x . Two intersections means $\Delta > 0$; the line is a *tangent* (touches) when $\Delta = 0$; no intersection when $\Delta < 0$; and "meets or touches" means $\Delta \geq 0$.

3. Simultaneous equations. If one equation is linear, substitute it into the other to get a quadratic in one variable. "Two distinct real solutions for x " means $\Delta > 0$.

Strategy: Count first, solve second

When a question asks *how many* solutions, or for which parameter values, you almost never need the solutions themselves. Reduce to one quadratic, write down Δ , and solve the inequality in the parameter. Always check the leading coefficient first: if it can be 0 (or the parameter cancels it), the equation may stop being quadratic.

Worked example: always positive

Question. Find the set of values of k for which $x^2 + (k-1)x + k + 2 > 0$ for every real x .

Solution. The graph is an upward parabola, so it stays above the axis exactly when $\Delta < 0$:

$$\Delta = (k-1)^2 - 4(k+2) = k^2 - 6k - 7 = (k-7)(k+1) < 0.$$

The product of two brackets is negative between the roots, so $-1 < k < 7$.

Examiner's eye. Test the answer with one value inside, say $k = 0$: $x^2 - x + 2$ has $\Delta = 1 - 8 < 0$. At the endpoints $k = -1$ or $k = 7$ the discriminant is 0 and the quadratic touches the axis, so it is *not* strictly positive. That is why the inequality is strict.

Worked example: a line touching a curve

Question. The line $y = 2x + c$ touches the curve $y = x^2 - 4x + 7$. Find c . Then find all c for which the line meets the curve at least once.

Solution. Set equal: $x^2 - 4x + 7 = 2x + c$, so $x^2 - 6x + (7-c) = 0$. Then $\Delta = 36 - 4(7-c) = 8 + 4c$. Touching needs $\Delta = 0$, so $c = -2$. Meeting at least once needs $\Delta \geq 0$, so $c \geq -2$.

Examiner's eye. A geometric check: the curve has minimum value 3 at $x = 2$, where its gradient is 0. The line has gradient 2, so it touches where $2x - 4 = 2$, i.e. at $x = 3$, $y = 4$. Then $c = 4 - 6 = -2$. Two methods, one answer.

Worked example: simultaneous equations, two solutions

Question. The simultaneous equations $x^2 + 3xy = 4$ and $x + y = a$ are to have two distinct real solutions for x . Find the set of values of a .

Solution. Substitute $y = a - x$: $x^2 + 3x(a - x) = 4$, so $-2x^2 + 3ax - 4 = 0$, i.e. $2x^2 - 3ax + 4 = 0$. Then $\Delta = 9a^2 - 32 > 0$, so $a^2 > \frac{32}{9}$:

$$a < -\frac{4\sqrt{2}}{3} \quad \text{or} \quad a > \frac{4\sqrt{2}}{3}.$$

Examiner's eye. The x^2 terms combined ($1 - 3 = -2$). Always simplify before you decide what the leading coefficient is. If they had cancelled completely the equation would be linear and give one solution, not two.

Common mistakes

- Using $\Delta \geq 0$ when the question says *distinct*, or $\Delta > 0$ when it says *at least one*. Read the wording, then decide whether the end-points are in.
- Forgetting that $a < 0$ flips "positive for all x ".
- Dividing the inequality $k^2 > c$ by k or writing $k > \pm\sqrt{c}$. Say " $k < -\sqrt{c}$ or $k > \sqrt{c}$ ".

Completing the square, and quadratics in disguise

Writing $ax^2 + bx + c = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$ shows the vertex, and therefore the minimum (if $a > 0$) or maximum (if $a < 0$). A square is never negative, so $(x - 3)^2 \geq 0$ with equality only at $x = 3$.

The TMUA loves to hide a quadratic. Whenever you see a repeated block, *give it a name*. If $t = 3^x$ then $9^x = t^2$; if $t = \cos x$ then $\sin^2 x = 1 - t^2$; if $t = x^2$ then $x^4 = t^2$. Two warnings: (i) t usually has a **restricted range** (for $t = 3^x$, $t > 0$; for $t = \cos x$, $-1 \leq t \leq 1$), so a vertex outside that range does not count; (ii) a function like $\frac{1}{q(x)}$ is largest where q is smallest, provided $q > 0$.

Worked example: a minimum in disguise

Question. Find the minimum value of $9^x - 6 \cdot 3^x + 10$.

Solution. Let $t = 3^x > 0$. Then the expression is $t^2 - 6t + 10 = (t - 3)^2 + 1$. The vertex $t = 3$ is allowed ($3^x = 3$ at $x = 1$), so the minimum is 1.

Worked example: the vertex is not allowed

Question. Find the minimum value of $4^x - 2^{x+2} + 7$ for $x \geq 0$.

Solution. Let $t = 2^x$, so $t \geq 1$ when $x \geq 0$. The expression is $t^2 - 4t + 7 = (t - 2)^2 + 3$. The vertex $t = 2$ is allowed ($x = 1$), so the minimum is 3. (If the question had said $x \leq 0$ then $0 < t \leq 1$, the vertex would be outside, and the minimum would be at $t = 1$: value 4.)

Worked example: reciprocal of a quadratic

Question. Find the largest value of $\frac{1}{x^2 - 4x + 9}$.

Solution. $x^2 - 4x + 9 = (x - 2)^2 + 5 \geq 5$, and it is always positive, so the reciprocal is largest where the denominator is smallest: $\frac{1}{5}$.

Strategy: rename the repeated block

If a block repeats (e.g. 2^x , $\sin x$, x^2), call it t , solve the quadratic in t , then ask: which values of t are *possible*, and how many x does each give? For example $\cos x = t$ has two solutions in $[0, 2\pi)$ for $-1 < t < 1$ but only one at $t = \pm 1$.

Sum and product of roots

If $ax^2 + bx + c = 0$ has roots α, β then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. This lets you evaluate symmetric expressions without ever finding the roots:

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta, \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}, \quad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta.$$

To build a new quadratic with roots p, q you only need $p + q$ and pq : it is $x^2 - (p + q)x + pq = 0$.

Worked example: roots of a transformed quadratic

Question. The roots of $x^2 - 6x + 4 = 0$ are α and β . Find a quadratic equation with roots $\sqrt{\alpha}$ and $\sqrt{\beta}$.

Solution. Here $\alpha + \beta = 6$ and $\alpha\beta = 4$, and both roots are positive (sum and product positive, $\Delta = 20 > 0$), so the square roots exist. The product is $\sqrt{\alpha}\sqrt{\beta} = \sqrt{4} = 2$. The sum satisfies $(\sqrt{\alpha} + \sqrt{\beta})^2 = \alpha + \beta + 2\sqrt{\alpha\beta} = 6 + 4 = 10$, so the sum is $\sqrt{10}$ (positive). The quadratic is $x^2 - \sqrt{10}x + 2 = 0$.

Examiner's eye. Squaring the sum is the move. The mistake to avoid is to say $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{\alpha + \beta}$: roots do not distribute over addition.

Inequalities, the modulus and bounds

Polynomial inequalities. Find the roots, mark them on a number line, and determine the sign in each region (check one test value, or use the leading coefficient for the far right). For *two* conditions that must both hold, solve each separately and take the overlap.

The modulus. $|u|$ is the distance of u from 0, so $|x - a| < r$ means $a - r < x < a + r$ ("x is within r of a "). For an equation with $|\cdot|$ on one side and an expression on the other, split into the two cases *and check each answer in the original*. A solution that makes the right-hand side negative must be rejected.

Bounds on products. If $a \leq x \leq b$ and $c \leq y \leq d$, the extremes of xy occur at the four "corners" ac , ad , bc , bd , because for fixed y the product is linear in x . Evaluate all four and compare. Do not assume the biggest numbers give the biggest product: two negatives multiply to a positive.

Worked example: two conditions together

Question. Find all x such that both $(x + 1)(x - 2)(3 - x) > 0$ and $x^2 < 4$.

Solution. First inequality: roots $-1, 2, 3$. For large x the expression is negative (the leading term is $-x^3$), so the signs from the right are $-, +, -, +$. It is positive for $x < -1$ and $2 < x < 3$. Second: $-2 < x < 2$. The overlap is $-2 < x < -1$.

Worked example: corners

Question. The real numbers x, y satisfy $|x - 1| \leq 3$ and $|y + 2| \leq 5$. Find the greatest possible value of xy , and of $|xy|$.

Solution. $-2 \leq x \leq 4$ and $-7 \leq y \leq 3$. The corner products are $(-2)(-7) = 14$, $(-2)(3) = -6$, $4(-7) = -28$, $4(3) = 12$. So the greatest xy is 14 and the greatest $|xy|$ is 28.

Worked example: a modulus equation with false roots

Question. Solve $|2x - 1| = x - 4$.

Solution. The right-hand side must be ≥ 0 , so any solution has $x \geq 4$. Case $2x - 1 \geq 0$: $2x - 1 = x - 4$ gives $x = -3$, not ≥ 4 , rejected. Case $2x - 1 < 0$: $1 - 2x = x - 4$ gives $x = \frac{5}{3}$, which fails $x \geq 4$. So there are **no solutions**.

Common mistakes

- Turning $x^2 > 9$ into $x > 3$. It is $x < -3$ or $x > 3$.
- Multiplying or dividing an inequality by an expression that can be negative. Rearrange to expression > 0 and use a sign diagram instead.
- Not checking modulus solutions, or forgetting that the strict/non-strict end-points of two conditions can differ.

Factor and remainder theorems

Factor theorem: $(x - a)$ is a factor of $p(x)$ if and only if $p(a) = 0$. **Remainder theorem:** the remainder when $p(x)$ is divided by $(x - a)$ is $p(a)$. For a divisor $(px - q)$ use $x = \frac{q}{p}$. These two facts avoid long division completely.

With two unknown coefficients, each fact about a factor or remainder gives a linear equation; two facts, two equations. When the question asks only for a *remainder*, you rarely need the unknowns individually: look for the combination you need.

Worked example: two factors, two unknowns

Question. $(x + 1)$ and $(x - 3)$ are factors of $2x^3 + px^2 + qx + 6$. Find p and q , and the remainder on division by $(x + 2)$.

Solution. $x = -1$: $-2 + p - q + 6 = 0$, so $p - q = -4$. $x = 3$: $54 + 9p + 3q + 6 = 0$, so $3p + q = -20$. Adding the two: $4p = -24$, $p = -6$, $q = -2$. The cubic is $2x^3 - 6x^2 - 2x + 6$. The remainder on division by $(x + 2)$ is the value at $x = -2$: $-16 - 24 + 4 + 6 = -30$.

Worked example: which assignment is best?

Question. $f(x) = x^3 + ax^2 + bx + c$, where a, b, c are 1, 2, 3 in some order. The remainder on division by $(x - 1)$ is R and on division by $(x + 1)$ is S . Find the greatest possible value of $R - S$.

Solution. $R = f(1) = 1 + a + b + c$ and $S = f(-1) = -1 + a - b + c$. Subtract: $R - S = 2 + 2b = 2(1 + b)$. Only b matters, and the largest it can be is 3, so the greatest value is 8.

Examiner's eye. Before trying all six orders, ask what survives in the difference $f(1) - f(-1)$. Even powers cancel, odd powers double. A whole family of TMUA remainder questions collapses this way.

Functions

A **function** sends each input in its *domain* to exactly one output; the set of outputs is the *range*. $fg(x) = f(g(x))$: apply g first. To find an inverse, write $y = f(x)$, make x the subject, then swap the letters; the domain and range swap too. The graph of f^{-1} is the reflection of $y = f(x)$ in $y = x$.

The TMUA often defines a function by a *rule* and asks you to use it cleverly. Write down the rule, then feed it the *smallest useful numbers*, chosen so that each side of the rule involves things you are given.

Worked example: an inverse and a self-inverse

Question. Let $f(x) = \frac{2x+3}{x-1}$ for $x \neq 1$. Find $f^{-1}(x)$ and the range of f . Is f its own inverse?

Solution. $y(x-1) = 2x+3 \Rightarrow x(y-2) = y+3 \Rightarrow x = \frac{y+3}{y-2}$, so $f^{-1}(x) = \frac{x+3}{x-2}$. The range is the domain of f^{-1} : all real $y \neq 2$. It is not self-inverse, because $f^{-1} \neq f$ (compare the numerators: $2x+3$ against $x+3$).

Worked example: a function defined by a rule

Question. A function h is defined on the positive integers by: $h(ab) = h(a)h(b)$ whenever ab is even, and $h(n) = n$ whenever n is odd. Given that $h(6) + h(10) - h(15) = 1$, find $h(2)$.

Solution. 15 is odd, so $h(15) = 15$. Use the rule with $6 = 2 \times 3$ (even): $h(6) = h(2)h(3) = 3h(2)$, since 3 is odd. Likewise $h(10) = h(2)h(5) = 5h(2)$. So $3h(2) + 5h(2) - 15 = 1$, giving $h(2) = 2$.

Surds and indices

Rationalise a denominator by multiplying by the conjugate: $\frac{1}{\sqrt{a}-\sqrt{b}} = \frac{\sqrt{a}+\sqrt{b}}{a-b}$. For indices, write every base as a power of one prime and compare exponents: $2^{3x} = 8^{y+2}$ means $3x = 3(y+2)$. Remember $(a^m)^n = a^{mn}$, $a^{-n} = \frac{1}{a^n}$, $a^{m/n} = \sqrt[n]{a^m}$, and that a^{b^c} means $a^{(b^c)}$.

Worked example: indices with two unknowns

Question. Given $2^x \cdot 4^y = 32$ and $\frac{3^x}{9^y} = 27$, find $x + y$.

Solution. First equation: $2^{x+2y} = 2^5$, so $x + 2y = 5$. Second: $3^{x-2y} = 3^3$, so $x - 2y = 3$. Adding, $2x = 8$, $x = 4$, then $y = \frac{1}{2}$. So $x + y = \frac{9}{2}$.

Worked example: a surd in a denominator

Question. Write $\frac{6}{\sqrt{7}-1}$ in the form $a + b\sqrt{7}$.

Solution. Multiply top and bottom by $\sqrt{7} + 1$: $\frac{6(\sqrt{7}+1)}{7-1} = \sqrt{7} + 1$. So $a = 1$, $b = 1$.

Habits that win marks on algebra questions

- **Read for the target.** Find the thing the question wants (a count, a sum, a set of values) and ask whether you can get it without solving everything.
- **Test a value.** Put a small number into the options. If $k = 0$ works in the original, delete every option that excludes 0. Do the same at the end-points.
- **Keep the leading coefficient honest.** Before you use Δ , check the equation really is quadratic and which way the parabola opens.
- **Check anything that squares, divides or takes a root.** Plug the answer back into the *original* equation.
- **Stop at two minutes.** If the algebra is ugly, circle the question and move on. The next one may take 40 seconds.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Find the set of values of k for which $x^2 - kx + 9 = 0$ has two distinct real roots.
2. The line $y = x + c$ is a tangent to the curve $y = x^2 - 5x + 10$. Find c .
3. Find the minimum value of $25^x - 4 \cdot 5^x + 9$.
4. $x^2 - 5x + 3 = 0$ has roots α, β . Find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$.
5. Solve $|x - 3| < 2x$.
6. $(x - 2)$ is a factor of $x^3 + px^2 - 4x - 12$. Find p and factorise.
7. $-3 \leq x \leq 2$ and $-1 \leq y \leq 4$. Find the greatest value of xy and of $|xy|$.
8. The remainder when $x^3 - 2x + k$ is divided by $(x + 1)$ is 4. Find k .

Answers to the self-check

1. $\Delta = k^2 - 36 > 0$, so $k < -6$ or $k > 6$.
2. $x^2 - 6x + 10 - c = 0$ has $\Delta = 36 - 4(10 - c) = 0$, so $c = 1$.
3. With $t = 5^x > 0$: $(t - 2)^2 + 5$, vertex allowed ($t = 2$). Minimum 5.
4. $\alpha + \beta = 5$, $\alpha\beta = 3$: $\alpha^2 + \beta^2 = 25 - 6 = 19$ and $\frac{5}{3}$.
5. If $x \geq 3$: $x - 3 < 2x$ is always true. If $x < 3$: $3 - x < 2x \Rightarrow x > 1$. So $x > 1$.
6. $8 + 4p - 8 - 12 = 0$, $p = 3$; $x^3 + 3x^2 - 4x - 12 = (x + 3)(x - 2)(x + 2)$.
7. Corners: 3, -12, -2, 8. Greatest $xy = 8$, greatest $|xy| = 12$.
8. $-1 + 2 + k = 4$, so $k = 3$.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.