

TMUA

Paper 1

Exponentials and logarithms

Free lesson notes

Covers	1.5 exponentials, logarithms, equations and graphs
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Exponentials and logarithms look like a topic about formulae, but on the TMUA they are mostly a topic about *structure*. An equation in 2^x , 4^x and 2^{x+3} is a quadratic in disguise. A pair of logarithm equations is a pair of ordinary equations in the unknowns $\log x$ and $\log y$. A “which is largest?” question is solved by raising everything to a common power. And a logarithm only exists for positive input, so about half of the traps are domain traps.

This lesson teaches the laws (briefly: you know them), then the moves that win marks: choose a common base, substitute to reveal a quadratic, rename $\log x$ as a variable, compare sizes without evaluating, and sketch before counting. There are twelve worked examples, each original and each verified by computer.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **Exponential equations that are quadratics in disguise:** 3^x , 9^x and $(\sqrt{3})^x$ in one equation, solved with $t = a^x$ (and $t > 0$).
- **Equations quadratic in $\log_2 x$,** including ones with $\log_x 2$, solved with $u = \log_2 x$ and the change-of-base rule $\log_x 2 = 1/\log_2 x$. Then check $x > 0$ and $x \neq 1$.
- **Simultaneous logarithm equations** such as one involving $\log(xy^2)$ and another involving the product $\log x \cdot \log y$: write $u = \log x$, $v = \log y$ and solve the resulting algebra.
- **Minimum or maximum of an exponential expression** such as $2^{2x} - 2^{x+k} + c$, found by substituting $t = 2^x$ and completing the square, with a parameter k to determine.
- **Which is largest?** Numbers in different forms (powers with different bases, logarithms) compared by raising to a common power or sandwiching between simple numbers.
- **Graphs:** identify or sketch 2^x , $\log_2 x$ and $\frac{2^x}{1+2^x}$ with their asymptotes, count solutions of $2^x = x^2$, or find the equation of a line through points with logarithmic coordinates.
- **Growth and decay** with a half-life, solved without a calculator by choosing numbers that are powers of 2, and **logarithm inequalities** where the domain matters.

Almost every one of these questions has a single crucial step: *write everything in a common form*. Look for the common base, the repeated block, or the common power before you do anything else.

Indices and logarithms: the laws and the domain

You know the laws. The skill is choosing which to use, and remembering what each assumes.

Key ideas

Indices: $a^m a^n = a^{m+n}$, $\frac{a^m}{a^n} = a^{m-n}$, $(a^m)^n = a^{mn}$, $a^{-n} = \frac{1}{a^n}$, $a^{1/n} = \sqrt[n]{a}$.

Logs: $\log_a b = c \iff a^c = b$ (with $a > 0$, $a \neq 1$, $b > 0$). $\log_a(xy) = \log_a x + \log_a y$, $\log_a \frac{x}{y} = \log_a x - \log_a y$,

$\log_a x^k = k \log_a x$, $\log_a a = 1$, $\log_a 1 = 0$.

Change of base: $\log_a b = \frac{\log_c b}{\log_c a}$, so in particular $\log_a b = \frac{1}{\log_b a}$.

Strategy: same base, then equate exponents

For an equation like $4^x = 8^{y-1}$, write both sides as powers of 2: $2^{2x} = 2^{3y-3}$, so $2x = 3y - 3$. A different base on each side is a sign that taking logarithms is needed; the same base on each side lets you simply equate exponents.

Strategy: the domain check is part of the answer

After combining logarithms you often get a quadratic with two roots. Substitute *each* root back into the *original* equation: a logarithm only exists for a positive argument, so a root that makes any original argument ≤ 0 must be rejected. Combining $\log_a(x+1) + \log_a(x-2)$ into a product silently *widens* the domain, which is how false roots appear.

Worked example: combine, solve, then check the domain

Question. Solve $\log_2(x+1) + \log_2(x-2) = 2$.

Solution. Both logarithms need $x+1 > 0$ and $x-2 > 0$, so $x > 2$. Combining: $\log_2((x+1)(x-2)) = 2$, so $(x+1)(x-2) = 4$, i.e. $x^2 - x - 6 = 0$ and $(x-3)(x+2) = 0$. The root $x = -2$ is not greater than 2 (indeed $\log_2(-1)$ does not exist), so it is rejected. Hence $x = 3$.

Examiner's eye. Verify the survivor: $\log_2 4 + \log_2 1 = 2 + 0 = 2$. The rejected root is the classic trap: $(-2+1)(-2-2) = 4$ does satisfy the combined equation, but not the original one.

Common mistakes

- Writing $\log(x+y) = \log x + \log y$. The law is for $\log(xy)$ only.
- Treating $\log_a b$ as $\log_b a$: they are reciprocals, not equal (unless both are ± 1).
- Forgetting that $\log_a x^k = k \log_a x$ needs $x > 0$: $\log(x^2) = 2 \log x$ only for $x > 0$ (for negative x it is $2 \log |x|$).
- Dropping the restriction $x \neq 1$ when the base of a logarithm is the unknown: $\log_x 2$ requires $x > 0$ and $x \neq 1$.

Exponential and logarithmic equations

There are three moves, in order of preference: (1) **common base** and equate exponents; (2) **substitute** $t = a^x$ (or $u = \log x$) to reveal a quadratic or a pair of simple equations; (3) **take logarithms** of both sides, when the bases cannot be matched.

Strategy: name the repeated block

If a^x appears in several forms, such as a^{2x} , a^{x+1} or $a^{x/2}$, write every term as a power of one block $t = a^{x/2}$ (the smallest convenient power) so that $a^x = t^2$, $a^{2x} = t^4$, and so on. Remember $t > 0$ for all real x , so a negative root of the quadratic is rejected.

Worked example: a surd base

Question. Solve $3^x - 4(\sqrt{3})^x + 3 = 0$.

Solution. Since $3^x = ((\sqrt{3})^x)^2$, let $t = (\sqrt{3})^x > 0$. Then $t^2 - 4t + 3 = 0$, so $(t-1)(t-3) = 0$ and $t = 1$ or $t = 3$. $(\sqrt{3})^x = 1$ gives $x = 0$; $(\sqrt{3})^x = 3 = (\sqrt{3})^2$ gives $x = 2$. Solutions: $x = 0$ and $x = 2$.

Examiner's eye. Both roots of the quadratic were positive, so both survive. In a variant such as $t^2 - 2t - 3 = 0$, the root $t = -1$ would have to be rejected: an exponential is never negative.

Worked example: an exponent identity

Question. The non-zero real numbers x, y, z satisfy $2^x = 3^y = 6^z$. Show that $\frac{1}{z} = \frac{1}{x} + \frac{1}{y}$.

Solution. Let the common value be N . Take natural logarithms: $x \ln 2 = y \ln 3 = z \ln 6 = L$. Here $L \neq 0$, since $x \neq 0$ and $\ln 2 \neq 0$. Then $\frac{1}{x} = \frac{\ln 2}{L}$, $\frac{1}{y} = \frac{\ln 3}{L}$ and $\frac{1}{z} = \frac{\ln 6}{L}$. Since $\ln 6 = \ln 2 + \ln 3$,

$$\frac{1}{z} = \frac{\ln 2 + \ln 3}{L} = \frac{1}{x} + \frac{1}{y}.$$

Examiner's eye. The trick is to give the common value a name (here L after taking logs) and then express every reciprocal over L . Any " $a^x = b^y = c^z$ " question is solved this way.

Worked example: quadratic in $\log_2 x$

Question. Solve $\log_2 x + 6 \log_x 2 = 5$.

Solution. We need $x > 0$ and $x \neq 1$. By change of base, $\log_x 2 = \frac{1}{\log_2 x}$. Let $u = \log_2 x$ (so $u \neq 0$): $u + \frac{6}{u} = 5$, so $u^2 - 5u + 6 = 0$, $(u - 2)(u - 3) = 0$. $u = 2$ gives $x = 4$; $u = 3$ gives $x = 8$. Both satisfy $x > 0$, $x \neq 1$.

Examiner's eye. Quick check of $x = 4$: $\log_2 4 = 2$ and $\log_4 2 = \frac{1}{2}$, so $2 + 6 \cdot \frac{1}{2} = 5$. The change-of-base step $\log_x 2 = 1/\log_2 x$ is the move to remember: it turns "logs in two different bases" into a single unknown u .

Worked example: simultaneous logarithm equations

Question. Find all pairs (x, y) of positive reals with $\log_3(x^2 y) = 7$ and $(\log_3 x)(\log_3 y) = 6$.

Solution. Let $u = \log_3 x$ and $v = \log_3 y$. The first equation becomes $2u + v = 7$ and the second $uv = 6$. Substitute $v = 7 - 2u$: $u(7 - 2u) = 6$, so $2u^2 - 7u + 6 = 0$, $(2u - 3)(u - 2) = 0$. $u = 2$: $v = 3$, so $x = 9$, $y = 27$. $u = \frac{3}{2}$: $v = 4$, so $x = 3^{3/2} = 3\sqrt{3}$, $y = 81$. There are two solutions: $(9, 27)$ and $(3\sqrt{3}, 81)$.

Examiner's eye. Check the second pair: $x^2 y = 27 \cdot 81 = 3^3 \cdot 3^4 = 3^7$ and $\frac{3}{2} \cdot 4 = 6$. Renaming $\log x$ and $\log y$ turned a logarithm problem into a factorising problem. Both u -values gave valid x and y since powers of 3 are always positive.

Common mistakes

- Solving $3^x = -2$ or log of a negative number: say "no solution" rather than forcing an answer.
- Rejecting a root that is fine, or keeping one that makes a logarithm's argument negative. Check every root in the original equation.
- Taking logs of a sum: $\ln(a^x + b) \neq x \ln a + \ln b$. You can only take logs of a product or power.

Minimising, maximising and comparing sizes

The expressions $a^{2x} - p a^x + q$ are quadratics in $t = a^x$, and the minimum is found by completing the square, *provided the vertex value of t is allowed* ($t > 0$).

Worked example: a minimum with a parameter

Question. Let $f(x) = 4^x - 2^{x+k} + 9$, where k is a constant. Find the minimum value of f in terms of k , the value of k for which the minimum is exactly 0, and the values of k for which $f(x) > 0$ for all x .

Solution. Let $t = 2^x > 0$. Since $2^{x+k} = 2^k \cdot t$, $f = t^2 - 2^k t + 9 = (t - 2^{k-1})^2 + 9 - 4^{k-1}$. The vertex $t = 2^{k-1}$ is always positive, so it is allowed, and the minimum value is $9 - 4^{k-1}$. The minimum is 0 when $4^{k-1} = 9$, i.e. $2^{2k-2} = 2^{2 \log_2 3}$, so $k = 1 + \log_2 3$. $f > 0$ for all x exactly when the minimum is positive: $4^{k-1} < 9$, i.e. $k < 1 + \log_2 3$.

Examiner's eye. Check with $k = 1 + \log_2 3$: then $2^k = 6$ and $f = t^2 - 6t + 9 = (t - 3)^2$, which touches 0 at $t = 3$ (so $x = \log_2 3$). The same condition appears as "discriminant < 0 ": $\Delta = 4^k - 36 < 0 \iff 2^k < 6$.

Comparing sizes. When numbers are given in different forms (powers with different bases, logarithms, trigonometric values) you cannot compare them by evaluating. Instead:

Strategy: common power, or sandwich

(1) If the numbers are powers such as 2^{300} , 3^{200} , 5^{100} , write each as a 100th power: $2^{300} = (2^3)^{100} = 8^{100}$ and $3^{200} = 9^{100}$, so only the bases 8, 9 and 5 need comparing. (2) If the numbers are logarithms, compare each with a simple benchmark such as $\frac{3}{2}$ by writing the inequality as a power: $\log_2 3 > \frac{3}{2} \iff 3 > 2^{3/2} \iff 9 > 8$. Squaring (or cubing) to clear fractions is safe when both sides are positive.

Worked example: which is largest?

Question. (a) Put 2^{300} , 3^{200} and 5^{100} in increasing order. (b) Which is larger, $\log_2 3$ or $\log_3 5$?

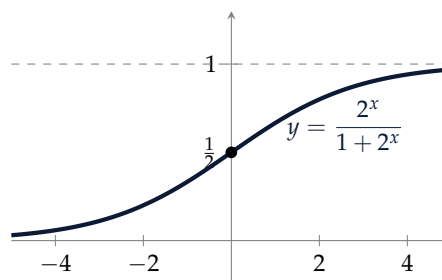
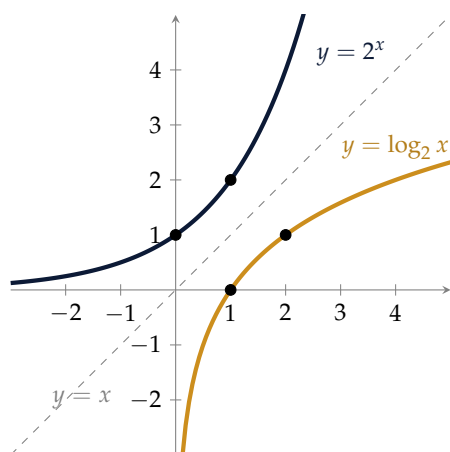
Solution. (a) Each exponent is a multiple of 100: $2^{300} = 8^{100}$, $3^{200} = 9^{100}$, $5^{100} = 5^{100}$. A larger base gives a larger power, so $5^{100} < 2^{300} < 3^{200}$.

(b) Compare both with $\frac{3}{2}$. $\log_2 3 > \frac{3}{2} \iff 3 > 2^{3/2} \iff 9 > 8$, true. $\log_3 5 < \frac{3}{2} \iff 5 < 3^{3/2} \iff 25 < 27$, true. So $\log_3 5 < \frac{3}{2} < \log_2 3$.

Examiner's eye. Such questions are built so that a tiny integer comparison ($9 > 8$, $25 < 27$) decides. When you see logarithms to different bases, ask: "which easy number sits between them?"

Graphs, asymptotes and counting solutions

Know the shape of $y = a^x$ for $a > 1$: it passes through $(0, 1)$, is always positive and increasing, and has the x -axis as an **asymptote** on the left. Its inverse $y = \log_a x$ is the reflection in the line $y = x$: it passes through $(1, 0)$, exists only for $x > 0$, is increasing, and has the y -axis as an asymptote.



The second graph is $y = \frac{2^x}{1 + 2^x}$, which turns up in real TMUA papers. Write it as $1 - \frac{1}{1 + 2^x}$. As $x \rightarrow -\infty$, $2^x \rightarrow 0$ and $y \rightarrow 0$; as $x \rightarrow \infty$, 2^x dominates and $y \rightarrow 1$. So the curve is trapped between the asymptotes $y = 0$ and $y = 1$, rises throughout, and passes through $(0, \frac{1}{2})$. Moreover $y(-x) = 1 - y(x)$, so the graph has rotational symmetry about $(0, \frac{1}{2})$.

Strategy: asymptotes first

To sketch a function of a^x , find the behaviour at both ends ($x \rightarrow \pm\infty$) and one point, usually $x = 0$. For $\frac{p}{q + r2^x}$ -type functions that is often enough to choose the right graph from five options.

Worked example: range and symmetry

Question. Let $f(x) = \frac{2^x}{1 + 2^x}$. Show that $f(x) + f(-x) = 1$ for all x , and find the range of f .

Solution. $f(-x) = \frac{2^{-x}}{1 + 2^{-x}} = \frac{1}{2^x + 1}$ (multiplying top and bottom by 2^x), so $f(x) + f(-x) = \frac{2^x + 1}{1 + 2^x} = 1$.

For the range, write $f(x) = 1 - \frac{1}{1 + 2^x}$. As 2^x takes all values in $(0, \infty)$, $\frac{1}{1 + 2^x}$ takes all values in $(0, 1)$, so f takes all values in $(0, 1)$. The range is $0 < f(x) < 1$.

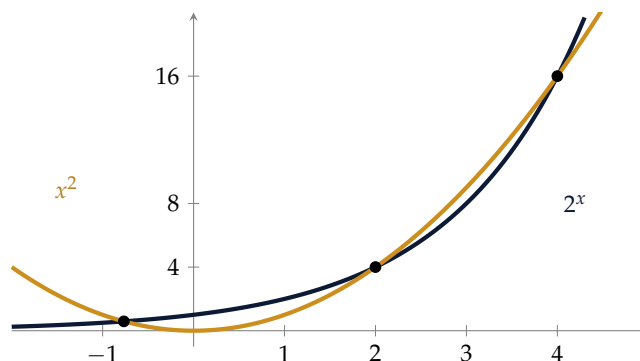
Examiner's eye. The end points are *not* attained: the curve approaches the asymptotes but never reaches them, so

the range is the open interval. If a question asks for the greatest value of f the answer is “there is none”.

Worked example: counting solutions with two curves

Question. How many real solutions does $2^x = x^2$ have?

Solution. Sketch $y = 2^x$ (positive, increasing, $y \rightarrow 0$ as $x \rightarrow -\infty$) and $y = x^2$ (a parabola through the origin). They meet at $x = 2$ ($4 = 4$) and $x = 4$ ($16 = 16$). For $x < 0$, 2^x falls from 1 to near 0 while x^2 falls from large to 0: at $x = -1$, $2^{-1} = \frac{1}{2} < 1 = x^2$, at $x = 0$, $1 > 0$, so they cross once between -1 and 0.



Between 2 and 4, x^2 is above 2^x (at $x = 3$, $9 > 8$); for $x > 4$ the exponential wins and keeps growing faster, so no more crossings. Total: 3 solutions.

Examiner's eye. The negative solution is easy to forget because it is not a nice number: it is enough to show by a sign change ($2^x - x^2$ is $-\frac{1}{2}$ at -1 and 1 at 0) that it exists. The sketch told you the count; the integers 2 and 4 are a free bonus.

Straight lines on logarithmic axes. If $y = ax^n$ then $\log y = \log a + n \log x$, so plotting $\log y$ against $\log x$ gives a straight line with gradient n and intercept $\log a$. If $y = ab^x$ then $\log y = \log a + x \log b$, so plotting $\log y$ against x gives a line with gradient $\log b$.

Worked example: a line through logarithmic points

Question. A graph of $\log_2 y$ against $\log_2 x$ is a straight line through the points $(2, 1)$ and $(4, 4)$. Find y in terms of x , and the value of y when $x = 64$.

Solution. The gradient is $\frac{4-1}{4-2} = \frac{3}{2}$, so $\log_2 y - 1 = \frac{3}{2}(\log_2 x - 2)$, i.e. $\log_2 y = \frac{3}{2} \log_2 x - 2$. Therefore $y = 2^{-2} x^{3/2} = \frac{x^{3/2}}{4}$.

For $x = 64$: $64^{3/2} = 8^3 = 512$, so $y = 128$.

Examiner's eye. The points $(2, 1)$ and $(4, 4)$ stand for $(x, y) = (4, 2)$ and $(16, 16)$: check $y = x^{3/2}/4$ gives $4^{3/2}/4 = 2$ and $16^{3/2}/4 = 16$. Always translate the labelled points back to the original variables.

Growth, decay and logarithm inequalities

Models of the form $N = N_0 a^{t/T}$ say the quantity multiplies by a every T units of time. For a half-life T , $N = N_0 \left(\frac{1}{2}\right)^{t/T} = N_0 2^{-t/T}$. Calculator-free questions use numbers that are powers of 2 (or 3), so every equation reduces to “equate the exponents”.

Worked example: two decaying samples

Question. Sample A has mass 384 g and its mass halves every 4 days. Sample B has mass 96 g and halves every 6 days. (a) After how many days does A have mass 3 g? (b) After how many days do the two samples have equal mass, and what is that mass?

Solution. $A = 384 \cdot 2^{-t/4}$ and $B = 96 \cdot 2^{-t/6}$. (a) $2^{-t/4} = \frac{3}{384} = \frac{1}{128} = 2^{-7}$, so $t = 28$ days. (b) $384 \cdot 2^{-t/4} = 96 \cdot 2^{-t/6}$ gives $4 = 2^{t/4 - t/6} = 2^{t/12}$, so $\frac{t}{12} = 2$ and $t = 24$ days. Then $A = 384 \cdot 2^{-6} = 6$ g, and $B = 96 \cdot 2^{-4} = 6$ g.

Examiner's eye. The method for any question “when are two exponentials equal” is to divide so the unknown ends up as a single power of one base, then equate exponents. Note that A starts heavier but decays faster, so the curves must cross: that is a useful sanity check on the sign of t .

Logarithm inequalities. Two rules. First, restrict to the domain before doing anything. Second, $\log_a x$ is increasing for $a > 1$ (so $\log_a x > \log_a y \iff x > y$) but *decreasing* for $0 < a < 1$ (the inequality flips).

Worked example: a logarithm inequality

Question. Solve $\log_3(x+6) > 1 + \log_3 x$.

Solution. The domain is $x > 0$ (which also makes $x+6 > 0$). Rearrange: $\log_3(x+6) - \log_3 x > 1$, so $\log_3 \frac{x+6}{x} > 1$, and since $\log_3$ is increasing, $\frac{x+6}{x} > 3$. As $x > 0$ we may multiply by x without reversing: $x+6 > 3x$, so $x < 3$. Combined with the domain: $0 < x < 3$.

Examiner's eye. At $x = 3$ both sides equal 2, so $x = 3$ is excluded (strict inequality). Sanity check at $x = 1$: $\log_3 7 > 1$ because $7 > 3$, so the inequality holds; at $x = 9$: $\log_3 15 < 3$ because $15 < 27$, whereas the right-hand side is $1 + \log_3 9 = 3$, so the inequality fails. The domain restriction $x > 0$ was essential: multiplying by x without knowing its sign would have been invalid.

Common mistakes

- Reversing the inequality for a base greater than 1, or forgetting to reverse it for a base between 0 and 1.
- Multiplying an inequality by an unknown expression without knowing its sign.
- Giving the solution of the rearranged inequality and forgetting the domain, or the other way round.

Habits that win marks

- **Common base first.** Before you take logs, see whether every term can be written as a power of the same number.
- **Substitute and restrict.** Let $t = a^x > 0$ or $u = \log x$, solve, then say what values the substitute can take.
- **Check every root** in the *original* logarithm equation: positive arguments, base positive and not 1.
- **Compare, don't compute.** Raise to a common power or sandwich between benchmark numbers ($1, \frac{3}{2}, 2$).
- **Sketch for counting.** Asymptotes and one point are enough to count solutions of $a^x = \text{polynomial}$.
- **Choose friendly numbers.** If your working produces an ugly decimal, you have probably missed a power of 2 or 3.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Evaluate $\log_4 8 + \log_9 27$.
2. Solve $9^x = 27^{x-1}$.
3. Solve $4^x - 5 \cdot 2^x + 4 = 0$.
4. Given $p = \log_2 3$, express $\log_{12} 18$ in terms of p .
5. Solve $\log_5 x + \log_5(x-4) = 1$.
6. Which is larger, 7^{30} or 2^{85} ?
7. How many real solutions has $2^x + x = 0$?
8. Solve $2 \log_2 x \leq \log_2(x+6)$.

Answers to the self-check

1. $\log_4 8 = \frac{3}{2}$ (as $4^{3/2} = 8$) and $\log_9 27 = \frac{3}{2}$ (as $9^{3/2} = 27$). Sum = 3.
2. $3^{2x} = 3^{3x-3}$, so $2x = 3x - 3$ and $x = 3$.
3. $t = 2^x$: $t^2 - 5t + 4 = (t - 1)(t - 4) = 0$, so $2^x = 1$ or 4 : $x = 0$ or $x = 2$.
4. $\log_{12} 18 = \frac{\log_2 18}{\log_2 12} = \frac{1 + 2p}{2 + p}$, since $18 = 2 \cdot 3^2$ and $12 = 2^2 \cdot 3$.
5. $x(x - 4) = 5$, so $x^2 - 4x - 5 = (x - 5)(x + 1) = 0$. The root $x = -1$ is outside the domain. Answer: $x = 5$.
6. $7^{30} = (7^6)^5$ and $2^{85} = (2^{17})^5$. $7^6 = 117649 < 131072 = 2^{17}$, so 2^{85} is larger.
7. $2^x + x$ is increasing (a sum of increasing functions), so it has at most one root. It is $-\frac{1}{2} < 0$ at $x = -1$ and $1 > 0$ at $x = 0$, so there is exactly 1 solution.
8. The domain is $x > 0$. Then $x^2 \leq x + 6$, so $(x - 3)(x + 2) \leq 0$, $-2 \leq x \leq 3$. Combined with $x > 0$: $0 < x \leq 3$.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.