

TMUA

Paper 1

Sequences and series

Free lesson notes

Covers	1.2 arithmetic and geometric sequences, sigma notation, recurrence, binomial
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Almost every TMUA Paper 1 contains a sequences question, and many contain two: one on arithmetic or geometric progressions, and one on the binomial (or multinomial) expansion. The arithmetic is rarely hard. The skill is turning a worded condition ("the sum of the first five terms equals the sum of the first eight") into an equation you can solve, and choosing the cheapest way to count when an expansion is too big to write out.

This lesson covers arithmetic and geometric sequences, infinite series, sigma notation, recurrences, and then the binomial and multinomial theorems with all the counting tricks the real papers use. There are eleven worked examples, each tied to a pattern that recurs, and each was checked by computer.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **A condition on sums:** the sum of the first m terms equals the sum of the first n terms (this gives a as a multiple of d), or two sums are in a given ratio.
- **Mixed progressions:** terms of a geometric progression that are also terms of an arithmetic progression, or an AP and a GP sharing a first term with conditions on their second and third terms.
- **Sum to infinity with a twist:** the series of squares or cubes of the terms, given the sum of the original series.
- **Recurrences that iterate:** a rule like $x_{n+1} = \sqrt{x_n}$, or a sequence defined by an integral. The answer comes from spotting the pattern in the first few terms.
- **Binomial coefficients:** the coefficient of x^k , the constant term of $(x^a + \frac{c}{x^b})^n$, an unknown constant chosen so two coefficients are equal.
- **Multinomials and sums of expansions:** the coefficient of x^2y^4 in $(1+x+y^2)^7$, or of x in $(1+x)^0 + (1+x)^1 + \dots + (1+x)^{80}$.
- **Two APs:** the numbers common to both, which form a new AP whose common difference is the lowest common multiple.

Arithmetic and geometric sequences

Arithmetic (add d): $u_n = a + (n-1)d$, and $S_n = \frac{n}{2}(2a + (n-1)d)$. **Geometric** (multiply by r): $u_n = ar^{n-1}$, $S_n = \frac{a(1-r^n)}{1-r}$, and when $|r| < 1$ the sum to infinity is $S_\infty = \frac{a}{1-r}$.

Three numbers x, y, z are in arithmetic progression exactly when $2y = x + z$ (the middle one is the mean) and in geometric progression exactly when $y^2 = xz$. Use these relationships when you only need a relation, not a, d or r themselves.

Strategy: write the condition, then simplify

Translate each sentence into one equation in a and d (or a and r) and simplify before solving. Equal sums of the first m and first n terms give a *ratio* $a : d$ and nothing more, which is exactly why such questions ask for a relationship, not a value.

Worked example: equal sums

Question. An arithmetic progression has first term a and common difference d . The sum of the first 6 terms equals the sum of the first 9 terms. Express a in terms of d , and find which term is zero.

Solution. $S_6 = 3(2a + 5d) = 6a + 15d$ and $S_9 = \frac{9}{2}(2a + 8d) = 9a + 36d$. Equal: $6a + 15d = 9a + 36d$, so $-3a = 21d$ and $a = -7d$. Then $u_n = a + (n-1)d = (n-8)d$, which is 0 at $n = 8$.

Examiner's eye. The terms from the seventh to the ninth must sum to zero: $u_7 + u_8 + u_9 = 3u_8 = 0$. So $u_8 = 0$ at once. Equal sums means the extra terms cancel, and a symmetric block of terms cancels about its middle.

Worked example: GP terms inside an AP

Question. The first, second and fourth terms of a geometric progression are also the first, fourth and seventh terms respectively of an arithmetic progression with common difference $d \neq 0$. Find the common ratio of the geometric progression.

Solution. Let the first term be a and GP ratio r . In the AP: $a, a + 3d, a + 6d$. So $ar = a + 3d$ and $ar^3 = a + 6d$. Subtract twice the first from the second: $ar^3 - 2ar = a - 2a = -a$, i.e. $r^3 - 2r = -1$ (taking $a \neq 0$), so $r^3 - 2r + 1 = 0$. $r = 1$ is a root: $(r-1)(r^2 + r - 1) = 0$. $r = 1$ would make $d = 0$, so $r^2 + r - 1 = 0$, giving $r = \frac{-1 \pm \sqrt{5}}{2}$.

Worked example: sum to infinity of powers

Question. A geometric series has sum to infinity 6. The series of the squares of its terms has sum to infinity 12. Find the sum to infinity of the series of cubes of its terms.

Solution. Let the first term be a , ratio r . Then $\frac{a}{1-r} = 6$ and the squares have first term a^2 , ratio r^2 : $\frac{a^2}{1-r^2} = 12$. Divide the second by the first: $\frac{a}{1+r} = 2$. So $a = 2(1+r)$ and $2(1+r) = 6(1-r)$, giving $r = \frac{1}{2}$, $a = 3$. The cubes have first term 27 and ratio $\frac{1}{8}$: sum $\frac{27}{7/8} = \frac{216}{7}$.

Common mistakes

- Using S_∞ when $|r| \geq 1$. It exists only for $|r| < 1$.
- Forgetting that the ratio of the squares series is r^2 (not r) and of the cubes series is r^3 .
- Counting from the wrong index: the n th term of an AP is $a + (n-1)d$, not $a + nd$.

Sigma notation, telescoping and recurrences

$\sum_{r=1}^n u_r$ means $u_1 + u_2 + \dots + u_n$. Constants come out of a sum and sums split: $\sum (3r + 2) = 3\sum r + 2n$. Standard results: $\sum_{r=1}^n r = \frac{n(n+1)}{2}$, and $\sum_{r=1}^n r^2 = \frac{n(n+1)(2n+1)}{6}$. **Telescoping:** if $u_r = f(r) - f(r+1)$ then almost everything cancels and $\sum_{r=1}^n u_r = f(1) - f(n+1)$.

A sequence from its sum. If S_n is the sum of the first n terms then $u_n = S_n - S_{n-1}$ for $n \geq 2$, and $u_1 = S_1$.

Recurrences. A rule $u_{n+1} = f(u_n)$ is explored by writing the first four or five terms. Look for a pattern (a repeat, a doubling, a power that doubles), prove or trust it, then jump to the term you need.

Worked example: a sequence from its sum

Question. $S_n = n^2 - 6n$ is the sum of the first n terms of a sequence. Find u_n , and the value of n for which S_n is smallest.

Solution. $u_1 = S_1 = -5$. For $n \geq 2$: $u_n = S_n - S_{n-1} = n^2 - 6n - (n-1)^2 + 6(n-1) = 2n - 7$. This also gives -5 at $n = 1$, so $u_n = 2n - 7$ for all n . S_n is smallest when we stop adding negative terms: $u_n < 0$ for $n \leq 3$, so $S_3 = -9$ is smallest ($S_n = (n-3)^2 - 9$).

Worked example: telescoping

Question. Find $\sum_{r=1}^{99} \frac{1}{r(r+1)}$.

Solution. $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$. The sum is $\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{99} - \frac{1}{100}\right) = 1 - \frac{1}{100} = \frac{99}{100}$.

Worked example: an iterating rule

Question. A sequence has $x_1 = 5$ and $x_{n+1} = \frac{x_n}{x_n - 1}$ for $n \geq 1$. Find x_{100} .

Solution. $x_2 = \frac{5}{4}$, $x_3 = \frac{5/4}{1/4} = 5$. So $x_3 = x_1$ and the sequence alternates $5, \frac{5}{4}, 5, \frac{5}{4}, \dots$ with period 2. Even-numbered terms are $\frac{5}{4}$, so $x_{100} = \frac{5}{4}$.

The binomial expansion

$(a + b)^n = \sum_{r=0}^n \binom{n}{r} a^{n-r} b^r$, where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. Useful rules: $\binom{n}{r} = \binom{n}{n-r}$, $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{n(n-1)}{2}$, and the row of Pascal's triangle sums to 2^n .

To find one coefficient, never expand. Write the general term, collect the power of the variable, and solve for r . In $(px^s + qx^t)^n$ the general term is $\binom{n}{r} (px^s)^{n-r} (qx^t)^r$, with the power of x equal to $s(n-r) + tr$.

Strategy: general term, then exponent

(1) Write $T_{r+1} = \binom{n}{r} (\text{first})^{n-r} (\text{second})^r$. (2) Add up the exponent of x and set it equal to the target (0 for a constant term). (3) Solve for r ; if r is not a whole number from 0 to n , the coefficient is 0. (4) Substitute back including *every* numerical factor, such as the 3 in $3x$ raised to a power.

Worked example: one coefficient

Question. Find the coefficient of x^3 in $(2 + 3x)^5$.

Solution. $T_{r+1} = \binom{5}{r} 2^{5-r} (3x)^r$. For $r = 3$: $\binom{5}{3} \cdot 2^2 \cdot 3^3 = 10 \cdot 4 \cdot 27 = 1080$.

Worked example: constant term

Question. Find the constant term in $\left(2x^2 - \frac{1}{x}\right)^9$.

Solution. $T_{r+1} = \binom{9}{r} (2x^2)^{9-r} \left(-\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} (-1)^r x^{18-2r-r}$. The exponent is $18 - 3r = 0$ at $r = 6$. Coefficient: $\binom{9}{6} \cdot 2^3 \cdot (-1)^6 = 84 \cdot 8 = 672$.

Worked example: matching two coefficients

Question. The non-zero constant k makes the coefficients of x^4 in $(1 + kx)^6$ and $(k + x^2)^7$ equal. Find k .

Solution. First: $\binom{6}{4} k^4 = 15k^4$. Second: the term in $x^4 = (x^2)^2$ has $r = 2$: $\binom{7}{2} k^5 = 21k^5$. Equal: $15k^4 = 21k^5$, so (as $k \neq 0$) $k = \frac{15}{21} = \frac{5}{7}$.

Common mistakes

- Forgetting the coefficients inside the brackets: in $(2 + 3x)^5$ the term for $r = 3$ includes 3^3 and 2^2 .
- Sign errors with $\left(-\frac{1}{x}\right)^r$: the sign is $(-1)^r$.
- Using r for the term number. The general term T_{r+1} starts at $r = 0$.

Multinomials and sums of expansions

For three or more terms, the coefficient of a particular product is a *counting* problem. In $(x_1 + x_2 + x_3)^n$ the term with $x_1^a x_2^b x_3^c$ ($a + b + c = n$) has coefficient $\frac{n!}{a! b! c!}$ (times any numerical multipliers). Reason: choose which a of the n brackets give x_1 , which b give x_2 , and the rest give x_3 .

Method for a trinomial. Match the powers of *each* variable. To get $x^2 y^4$ from $(1 + x + y^2)^7$, you need x twice and y^2 twice: that is two brackets giving x , two giving y^2 , three giving 1. The coefficient is $\frac{7!}{2! 2! 3!} = 210$.

Sums of expansions. The coefficient of x^m in $(1 + x)^k$ is $\binom{k}{m}$, so in $(1 + x)^m + (1 + x)^{m+1} + \dots + (1 + x)^N$ it is $\binom{m}{m} + \binom{m+1}{m} + \dots + \binom{N}{m} = \binom{N+1}{m+1}$ (the “hockey-stick” identity). Terms with $k < m$ contribute nothing.

Worked example: a trinomial coefficient

Question. Find the coefficient of $x^3 y^2$ in $(2 + x + y)^8$, and of $x^3 y^2$ in $(1 + x + y^2)^6$.

Solution. For $(2 + x + y)^8$: need x three times, y twice, and the constant 2 three times. Coefficient: $\frac{8!}{3! 2! 3!} \cdot 2^3 = 560 \cdot 8 = 4480$. For $(1 + x + y^2)^6$: $x^3 y^2$ needs x three times, y^2 once, 1 twice: $\frac{6!}{3! 1! 2!} = 60$.

Worked example: add up the expansions

Question. Find the coefficient of x^2 in $(1 + x)^2 + (1 + x)^3 + \dots + (1 + x)^{30}$.

Solution. Each $(1 + x)^k$ contributes $\binom{k}{2}$, so the answer is $\sum_{k=2}^{30} \binom{k}{2} = \binom{31}{3} = \frac{31 \cdot 30 \cdot 29}{6} = 4495$.

Worked example: a product of expansions

Question. Find the coefficient of x^4 in $(1 + x)^3 (1 + x + x^2 + x^3)^2$ without multiplying everything out.

Solution. The square $(1 + x + x^2 + x^3)^2$ has coefficients 1, 2, 3, 4, 3, 2, 1 for $x^0, \dots, x^6$ (they count the ways to make each power from two powers ≤ 3). Now pair each term of $(1 + x)^3 = 1 + 3x + 3x^2 + x^3$ with the term of the other factor that brings the total power to 4: $1 \cdot 3$ (from x^0 and x^4), $3 \cdot 4$ (from x^1 and x^3), $3 \cdot 3$ (from x^2 and x^2), $1 \cdot 2$ (from x^3 and x^1). The total is $3 + 12 + 9 + 2 = 26$.

Strategy: count arrangements, don't expand

For $(a + b + c)^n$, a term like $a^i b^j c^k$ has coefficient $\frac{n!}{i! j! k!}$. Write down which brackets contribute which factor, make sure the exponents on *every* variable match the target, and multiply by any numerical constants raised to the right powers.

Common terms of two progressions, and remainders

If two arithmetic sequences share some terms, the shared terms themselves form an arithmetic sequence. Find the first shared term (by listing a few terms of each, or by solving $a_1 + d_1 m = a_2 + d_2 n$), and the common difference of the shared sequence is $\text{lcm}(d_1, d_2)$. A remainder question then follows from this formula and modular arithmetic.

Worked example: shared terms

Question. Sequence A is 4, 9, 14, ... (first term 4, difference 5). Sequence B is 2, 8, 14, ... (first term 2, difference 6). Find the 15th number that appears in both, and its remainder on division by 7.

Solution. Both contain 14. The shared terms have difference $\text{lcm}(5, 6) = 30$, so the shared sequence is 14, 44, 74, ... and the 15th is $14 + 14 \times 30 = 434$. $434 = 7 \times 62$, so the remainder is 0.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. An arithmetic progression has $u_3 = 11$ and $u_7 = 27$. Find u_{20} .
2. The sum of the first 4 terms of an AP equals the sum of the first 10 terms. Find a in terms of d .
3. A geometric series has first term 12 and sum to infinity 18. Find its common ratio and the sum to infinity of the series of squares.
4. Find the coefficient of x^2 in $(1 - 2x)^6$.
5. Find the constant term in $\left(x + \frac{2}{x}\right)^6$.
6. Find the coefficient of xy^2 in $(1 + x + y)^5$.
7. Find the coefficient of x^3 in $(1 + x)^3 + (1 + x)^4 + \cdots + (1 + x)^{10}$.
8. $S_n = 2^n - 1$ is the sum of the first n terms. Find u_n .

Answers to the self-check

1. $4d = 16, d = 4, a = 3, u_{20} = 3 + 19 \times 4 = 79$.
2. $2(2a + 3d) = 5(2a + 9d)$, so $-6a = 39d$ and $a = -\frac{13}{2}d$.
3. $r = \frac{1}{3}$; squares: first term 144, ratio $\frac{1}{9}$: $\frac{144}{8/9} = 162$.
4. $\binom{6}{2} \cdot 4 = 60$.
5. $\binom{6}{3} \cdot 2^3 = 160$.
6. $\frac{5!}{1!2!2!} = 30$.
7. $\binom{11}{4} = 330$.
8. $u_n = 2^n - 2^{n-1} = 2^{n-1}$ (and $u_1 = 1$ fits).

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.