

TMUA

Paper 1

Trigonometry

Free lesson notes

Covers	1.4 trigonometric functions, identities, equations, triangles
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Trigonometry on the TMUA is almost never “find $\sin 37^\circ$ ”. It asks you to *count* solutions, to find a sign or an exact value from very little information, to spot that a long sum collapses in pairs, or to decide how many triangles fit some data. All of it is done without a calculator, in radians, and most of it rewards a quick sketch of $\sin$, $\cos$ or $\tan$ far more than a clever formula.

This lesson builds the picture first (radians, exact values, graphs and symmetry), then the equation-solving moves (factorising, quadratics in $\sin$ or $\cos$, the double angle formulae, squaring and adding), then triangles including the ambiguous case, and finally inequalities. You will see twelve worked examples, each original and each verified by computer.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **Exact values and signs:** given $\tan \theta$ and the quadrant, find $\sin \theta$, $\cos \theta$ or $\sin 2\theta$ exactly. A picture of a right-angled triangle and the “which quadrant” sign test do it in seconds.
- **How many solutions?** An equation mixing x with $\sin x$ or $\cos x$ (for example $\sin x = x/10$), or $\cos kx = c$ on a stated interval. Sketch both sides and count crossings.
- **Quadratics in $\sin x$ or $\cos x$,** including versions with $|\cos x|$, solved by substitution and then *checking each value is attainable* ($-1 \leq t \leq 1$).
- **Sums that collapse:** $\sin^2 0^\circ + \sin^2 1^\circ + \dots + \sin^2 90^\circ$ by pairing θ with $90^\circ - \theta$.
- **Symmetry of a transformed graph:** the line of symmetry or the period of $a \sin(bx + c) + d$.
- **Simultaneous trig equations,** solved by squaring and adding to use $\sin^2 + \cos^2 = 1$.
- **Triangles:** sine and cosine rules, area $\frac{1}{2}ab \sin C$, and the *ambiguous case* (two triangles, ratio of their areas, when is the triangle unique).
- **Comparing sizes and inequalities:** which of $\sin 1$, $\cos 1$, $\tan 1$ is smallest; for which x is $\tan x > \sin 2x$.

The calculator trap: the TMUA has none, but if you practise with one make sure it is in *radians*. A surprising number of lost marks in practice come from $\sin 2$ read in degrees.

Radians, exact values and signs

Radians. π radians = 180° , so 1 radian $\approx 57^\circ$. Always use radians unless the question says degrees. For a circle of radius r and an angle θ radians:

$$\text{arc length} = r\theta, \quad \text{sector area} = \frac{1}{2}r^2\theta.$$

Learn these exact values (they come from the 30-60-90 and 45-45-90 triangles):

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin \theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos \theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan \theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–

Key ideas

$$\sin^2 \theta + \cos^2 \theta = 1 \quad \tan \theta = \frac{\sin \theta}{\cos \theta} \quad 1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$$

Other angles by symmetry: $\sin(\pi - \theta) = \sin \theta$, $\cos(\pi - \theta) = -\cos \theta$, $\sin(\frac{\pi}{2} - \theta) = \cos \theta$, $\cos(-\theta) = \cos \theta$, $\sin(-\theta) = -\sin \theta$, $\tan(\theta + \pi) = \tan \theta$.

Signs by quadrant ("all, sin, tan, cos"): first quadrant all positive; second only sin; third only tan; fourth only cos.

Strategy: draw the triangle, then fix the sign

If you are given one ratio, say $\tan \theta = -2$, draw a right-angled triangle with the *unsigned* sides (2 and 1, hypotenuse $\sqrt{5}$), read off the unsigned sin and cos, and only then use the quadrant to give each its sign. This is quicker and safer than solving with identities and a $\pm$ square root.

Worked example: exact values from a tangent

Question. Given $\tan \theta = -2$ and $\frac{\pi}{2} < \theta < \pi$, find the exact values of $\cos \theta$, $\sin \theta$ and $\sin 2\theta$.

Solution. A triangle with opposite 2 and adjacent 1 has hypotenuse $\sqrt{5}$. So the unsigned values are $\sin = \frac{2}{\sqrt{5}}$ and $\cos = \frac{1}{\sqrt{5}}$. In the second quadrant sin is positive and cos is negative, so

$$\sin \theta = \frac{2}{\sqrt{5}}, \quad \cos \theta = -\frac{1}{\sqrt{5}}, \quad \sin 2\theta = 2 \sin \theta \cos \theta = -\frac{4}{5}.$$

Examiner's eye. Test it: $\tan \theta = \sin \theta / \cos \theta = \frac{2/\sqrt{5}}{-1/\sqrt{5}} = -2$. The negative sign of $\sin 2\theta$ is no accident: 2θ lies between π and 2π , where sine is negative.

Worked example: comparing sizes without a calculator

Question. Put $\sin 1$, $\cos 1$ and $\tan 1$ (angles in radians) in increasing order. Which is larger, $\sin 2$ or $\sin 3$?

Solution. Since $\pi \approx 3.14$, we have $\frac{\pi}{4} < 0.8 < 1 < \frac{\pi}{2}$. So 1 radian lies in $(\frac{\pi}{4}, \frac{\pi}{2})$, where $\sin > \cos > 0$ and $\tan > 1$ (because $\tan \frac{\pi}{4} = 1$ and $\tan$ is increasing). Also $\sin 1 < 1$. Therefore

$$\cos 1 < \sin 1 < 1 < \tan 1.$$

For the second part, $\frac{\pi}{2} < 2 < 3 < \pi$ (as $\pi > 3.14$), and $\sin$ is *decreasing* on $(\frac{\pi}{2}, \pi)$, so $\sin 2 > \sin 3$.

Examiner's eye. The tools here were: benchmark angles ($\frac{\pi}{4}$, $\frac{\pi}{2}$, π) and monotonicity on an interval. The most common wrong answer treats 2 and 3 as if they were degrees, or assumes sine is increasing.

Worked example: a sum that collapses in pairs

Question. Find the exact value of $S = \sin^2 5^\circ + \sin^2 10^\circ + \sin^2 15^\circ + \dots + \sin^2 85^\circ$ (angles in steps of 5°).

Solution. There are 17 terms, $\sin^2(5k^\circ)$ for $k = 1, \dots, 17$. Pair k with $18 - k$: since $\sin(90^\circ - \theta) = \cos \theta$,

$$\sin^2(5k^\circ) + \sin^2(90^\circ - 5k^\circ) = \sin^2(5k^\circ) + \cos^2(5k^\circ) = 1.$$

That gives 8 pairs, $k = 1, \dots, 8$ with $17, \dots, 10$. The middle term $k = 9$ is left over: $\sin^2 45^\circ = \frac{1}{2}$. So $S = 8 + \frac{1}{2} = \frac{17}{2}$.

Examiner's eye. Count the terms before pairing. With an odd number of terms there is always a lonely middle term, and it is exactly the self-paired angle (45° here, where $\sin = \cos$). A quick check is $S = \frac{17}{2}$: the average term is $\frac{1}{2}$, as it should be.

Common mistakes

- Mixing degrees and radians, or leaving a calculator in the wrong mode.
- Forgetting the sign: $\cos \frac{2\pi}{3} = -\frac{1}{2}$, not $+\frac{1}{2}$. Name the quadrant every time.
- Writing $\sin^2 \theta = \sin \theta^2$ or $\sin(2\theta) = 2 \sin \theta$. The first is a power of the value; the second is false.
- Believing $\sin \theta = \frac{1}{2}$ means $\theta = \frac{\pi}{6}$ only: there is always a second angle $\pi - \theta$ in $[0, 2\pi)$.

Graphs, symmetry and counting solutions

Keep the three graphs in your head. $y = \sin x$ and $y = \cos x$ both oscillate between -1 and 1 with period 2π (cosine is sine shifted left by $\frac{\pi}{2}$). $y = \tan x$ has period π and vertical asymptotes at $x = \frac{\pi}{2} + n\pi$. Both sin and cos equal a given c with $|c| < 1$ **exactly twice per period**.

Transformations. For $y = a \sin(bx + c) + d$: the amplitude is $|a|$, the midline is $y = d$, the period is $\frac{2\pi}{|b|}$, and the graph is shifted horizontally by $-c/b$. The graph has **lines of symmetry** through every maximum and minimum (where $bx + c = \frac{\pi}{2} + n\pi$) and **point (rotational) symmetry** about every midline crossing.

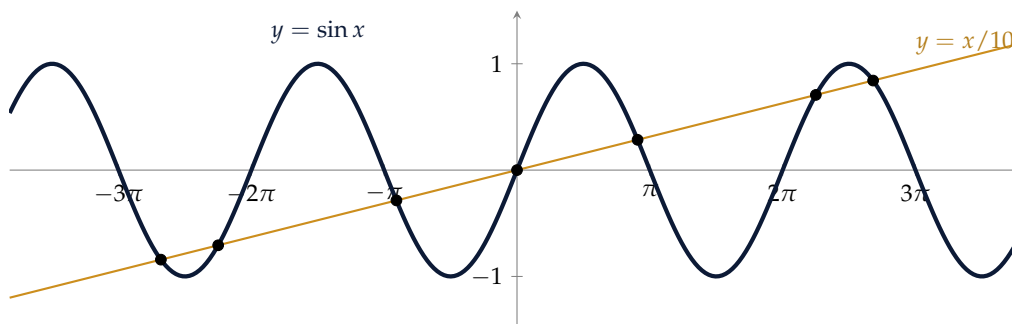
Strategy: stretching multiplies the count

If $\cos x = k$ has two solutions on an interval of length one period, then $\cos(3x) = k$ has $3 \times 2 = 6$ solutions on the same interval, because $3x$ sweeps through three periods. Replacing x by $x/2$ halves the number of periods swept. But take care at the end points of an interval, and remember $\cos^2 x = k^2$ means $\cos x = \pm k$: that is *two* equations.

Worked example: counting by sketching

Question. How many real solutions does $\sin x = \frac{x}{10}$ have?

Solution. Sketch $y = \sin x$ and the line $y = \frac{x}{10}$ (figure below). The line passes through the origin, so $x = 0$ is a solution. Both graphs are odd, so the solutions are symmetric about 0 and we count $x > 0$ and double.



For $x > 0$, the sine is above the line near 0 (slope $1 > \frac{1}{10}$). On the first hump $(0, \pi)$ the sine comes back down to 0 while the line is positive, so there is *one* more crossing. On the second hump $(2\pi, 3\pi)$ the sine climbs to 1, above the line (whose value there is under $\frac{3\pi}{10} < 1$), so it crosses *twice*. The next hump begins at $4\pi > 10$, where the line is already above 1: no more crossings. In fact for $x \geq 10$ the line is at least 1, so there are none. So $1 + 2 = 3$ solutions for $x > 0$, the same number for $x < 0$, and $x = 0$: a total of 7.

Examiner's eye. The deciding facts were the inequalities $3\pi < 10 < 4\pi$, which tell you which humps can reach the line. Always ask where the line leaves the band $-1 \leq y \leq 1$: beyond that point no solutions are possible, so you only need to examine the humps before it.

Worked example: line of symmetry of a transformed sine

Question. The graph of $y = 3 \sin(2x - \frac{\pi}{3}) + 1$ is symmetric about the line $x = a$. Find the smallest positive value of a , and the number of such lines with $0 \leq x \leq 2\pi$.

Solution. The lines of symmetry pass through the maxima and minima, where $\sin(2x - \frac{\pi}{3}) = \pm 1$, i.e. $2x - \frac{\pi}{3} = \frac{\pi}{2} + n\pi$. So

$$x = \frac{5\pi}{12} + \frac{n\pi}{2}, \quad n \in \mathbb{Z}.$$

For $n = -1$ the value is $-\frac{\pi}{12} < 0$, so the smallest positive line is $a = \frac{5\pi}{12}$. For $0 \leq x \leq 2\pi$: $n = 0, 1, 2, 3$ give $\frac{5\pi}{12}, \frac{11\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}$, and $n = 4$ gives $\frac{29\pi}{12} > 2\pi$. So there are 4 lines.

Examiner's eye. Period is π , so a maximum and the next minimum are $\frac{\pi}{2}$ apart: that is the spacing of the lines. The $+1$ and the factor 3 do not move any line of symmetry; only the b and c inside the bracket do.

Worked example: how stretching changes the count

Question. Let $0 < k < 1$. The equation $\cos x = k$ has exactly two solutions with $0 \leq x < 2\pi$. How many solutions in the same interval do (a) $\cos 3x = k$, (b) $\cos^2 x = k^2$, (c) $\cos \frac{x}{2} = k$ have?

Solution. (a) $3x$ runs through $[0, 6\pi)$, which is three periods, and each period has two solutions: **6**. (b) $\cos^2 x = k^2$ means $\cos x = k$ or $\cos x = -k$, two solutions each: **4**. (c) $\frac{x}{2}$ runs through $[0, \pi)$, half a period, and $\cos$ is strictly decreasing from 1 to -1 there, so it takes the value k exactly once: **1**.

Examiner's eye. Part (c) is the trap: "two solutions per period" is true for a *full* period only. A half period of cosine, from a maximum to a minimum, hits each value once.

Common mistakes

- Counting solutions for $\sin(2x) = c$ on $[0, 2\pi)$ as 2 because "sine takes each value twice". It is 4.
- Including or dropping the end points of an interval. Always check whether a solution sits exactly at 0 or 2π .
- Sketching the line $y = x/10$ too steep or too shallow. Mark where it reaches 1 ($x = 10$) and compare with the multiples of π .

Solving trigonometric equations

The core idea is to reduce to *one* trigonometric function of *one* angle. Use $\sin^2 x = 1 - \cos^2 x$ (or the reverse) to turn mixed equations into a quadratic, and the double-angle formulae to remove $2x$:

Key ideas

$$\sin 2x = 2 \sin x \cos x \quad \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$$

Strategy: factorise, never divide

If $\sin x$ or $\cos x$ appears as a factor on both sides, *move everything to one side and factorise*. Dividing by $\sin x$ silently throws away the solutions with $\sin x = 0$. The same goes for $\tan x = \dots$: remember $\tan x$ is undefined at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, so those cannot be solutions.

Strategy: substitute, then check attainable values

Put $t = \sin x$ (or $\cos x$) and solve the quadratic in t . Discard any root with $|t| > 1$: it gives no x . Then count the angles: $t = \pm 1$ gives *one* solution per period, any other t with $|t| < 1$ gives *two*.

Worked example: a quadratic in sine

Question. Solve $2 \cos^2 x + 3 \sin x = 3$ for $0 \leq x < 2\pi$.

Solution. Replace $\cos^2 x$ by $1 - \sin^2 x$: $2 - 2 \sin^2 x + 3 \sin x = 3$, so $2 \sin^2 x - 3 \sin x + 1 = 0$, i.e. $(2 \sin x - 1)(\sin x - 1) = 0$. $\sin x = \frac{1}{2}$ gives $x = \frac{\pi}{6}, \frac{5\pi}{6}$; $\sin x = 1$ gives $x = \frac{\pi}{2}$ (one solution only). So there are 3 solutions: $\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.

Examiner's eye. The repeated-looking root $\sin x = 1$ touches the top of the curve and contributes one solution, not two. Check one value: at $x = \frac{\pi}{6}$, $2 \cdot \frac{3}{4} + 3 \cdot \frac{1}{2} = 3$.

Worked example: a modulus inside a quadratic

Question. How many solutions does $4 \cos^2 x - 8|\cos x| + 3 = 0$ have for $0 \leq x < 2\pi$? How many does $4 \cos^2 x - 8 \cos x + 3 = 0$ have?

Solution. Since $\cos^2 x = |\cos x|^2$, let $t = |\cos x|$, with $0 \leq t \leq 1$. Then $4t^2 - 8t + 3 = (2t - 1)(2t - 3) = 0$, so $t = \frac{1}{2}$ or $t = \frac{3}{2}$. The second is impossible. So $|\cos x| = \frac{1}{2}$, i.e. $\cos x = \pm \frac{1}{2}$: $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$. That is **4** solutions. Without the modulus, $\cos x = t$ with the same roots, so only $\cos x = \frac{1}{2}$ is allowed (not $-\frac{1}{2}$): $x = \frac{\pi}{3}, \frac{5\pi}{3}$, which is **2** solutions.

Examiner's eye. The modulus doubles the count here because it admits both signs of cosine. Always ask what values the substituted variable can actually take: here $t \geq 0$ in the first version and $-1 \leq t \leq 1$ in the second.

Worked example: square and add

Question. Real numbers x and y satisfy $\sin x = 2 \sin y$ and $2 \cos x = \cos y$. Find $\tan^2 y$.

Solution. We want to remove x . Since $\sin^2 x + \cos^2 x = 1$, substitute $\sin x = 2 \sin y$ and $\cos x = \frac{1}{2} \cos y$:

$$4 \sin^2 y + \frac{1}{4} \cos^2 y = 1 \implies 16 \sin^2 y + \cos^2 y = 4.$$

Using $\cos^2 y = 1 - \sin^2 y$ gives $15 \sin^2 y + 1 = 4$, so $\sin^2 y = \frac{1}{5}$, $\cos^2 y = \frac{4}{5}$ and $\tan^2 y = \frac{1}{4}$.

Examiner's eye. **Square, then add** is the signal: when two equations each contain $\sin$ and $\cos$ of the *same* angle, squaring and adding uses $\sin^2 + \cos^2 = 1$ to eliminate that angle. Check existence: with $\tan y = \frac{1}{2}$, $\sin x = \frac{2}{\sqrt{5}}$ and $\cos x = \frac{1}{\sqrt{5}}$ are consistent, so such x, y really exist.

Common mistakes

- Dividing both sides of $\tan x = 2 \sin x$ by $\sin x$: you lose $x = 0$ and $x = \pi$. (Solve $\sin x(1 - 2 \cos x) = 0$ after multiplying by $\cos x$.)
- Squaring an equation and not checking: squaring can introduce solutions that fail the original.
- Giving one solution of $\sin x = c$ when the interval has two, or forgetting that $\sin x = -1$, $\cos x = -1$ and so on give a *single* solution per period.

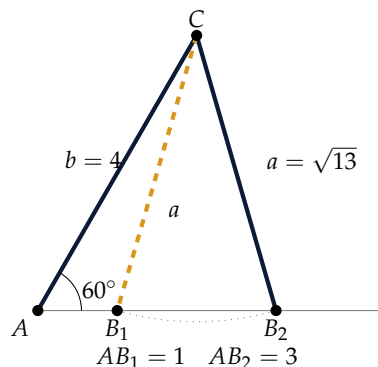
Triangles: sine rule, cosine rule and the ambiguous case

For a triangle with sides a, b, c opposite angles A, B, C :

Key ideas

sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$ area $= \frac{1}{2}ab \sin C$

Which rule? Two angles and a side: sine rule. Two sides and the angle *between* them, or all three sides: cosine rule. Two sides and an angle that is *not* between them (SSA): the **ambiguous case**. There the sine rule gives $\sin B$, and B could be acute or its supplement $180^\circ - B$, so there may be two triangles. A neater view is to use the cosine rule as a *quadratic in the unknown side*: the number of positive roots is the number of triangles.



Strategy: ambiguous case: a quadratic in the missing side

With the angle A , the side b next to it, and the side a opposite it, the cosine rule $a^2 = b^2 + c^2 - 2bc \cos A$ is a quadratic in c . Count its *positive* roots: two roots means two triangles, and the product of the roots shows which is bigger. Since both triangles share A and b , their areas $\frac{1}{2}bc \sin A$ are in the ratio of their c -values.

Worked example: cosine rule and an exact area

Question. A triangle has sides 4, 5 and 6. Find $\cos$ of its smallest angle and the exact area.

Solution. The smallest angle A is opposite the shortest side, $a = 4$: $\cos A = \frac{5^2 + 6^2 - 4^2}{2 \cdot 5 \cdot 6} = \frac{45}{60} = \frac{3}{4}$. Then

$\sin A = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$ (positive, since A is an angle of a triangle), and the area is $\frac{1}{2} \cdot 5 \cdot 6 \cdot \frac{\sqrt{7}}{4} = \frac{15\sqrt{7}}{4}$.

Examiner's eye. Check with Heron's formula: $s = \frac{15}{2}$ and the area squared is $\frac{15}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} = \frac{1575}{16} = \frac{225 \cdot 7}{16}$. Matches. The smallest angle always faces the shortest side, which saves you computing all three angles.

Worked example: the ambiguous case

Question. In triangle ABC , $\angle A = 60^\circ$, $AC = 4$ and $BC = \sqrt{13}$. Show there are two possible triangles, find the ratio of their areas, and find all values of $a = BC$ (with $A = 60^\circ$, $b = 4$ fixed) for which the triangle is unique.

Solution. Let $c = AB$. The cosine rule: $13 = 16 + c^2 - 2 \cdot 4 \cdot c \cos 60^\circ$, so $c^2 - 4c + 3 = 0$, $(c - 1)(c - 3) = 0$. Both roots are positive: $c = 1$ or $c = 3$, two triangles.

Both have the same $\frac{1}{2}bc \sin A$ form, so the ratio of areas is $3 : 1$.

For general a : $a^2 = 16 + c^2 - 4c$, i.e. $c^2 - 4c + (16 - a^2) = 0$. The discriminant is $16 - 4(16 - a^2) = 4a^2 - 48$ and the product of roots is $16 - a^2$.

- $a < 2\sqrt{3}$: no real roots, no triangle;
- $a = 2\sqrt{3}$: one root, $c = 2$ (right angle at B): one triangle;
- $2\sqrt{3} < a < 4$: two positive roots (their product $16 - a^2$ is positive, their sum is 4): two triangles;
- $a \geq 4$: product ≤ 0 , so exactly one positive root: one triangle.

Examiner's eye. In words: if the side opposite the given angle is at least as long as the side next to it ($a \geq b$), the swinging arm cannot reach the base on both sides of the foot of the perpendicular, so the triangle is unique. If $b \sin A < a < b$ you get two triangles, and $a = b \sin A$ is the tangent (right-angled) case.

Common mistakes

- Assuming SSA always gives exactly one triangle, or always two. Check a against $b \sin A$ and b .
- Using the sine rule to find an obtuse angle: the calculator (or arcsine) always gives the acute one. Think about the supplement.
- Using $\frac{1}{2}ab \sin C$ with C not the angle *between* a and b .

Inequalities from identities and graphs

To compare trigonometric functions on an interval you can reduce to a single function with an identity, or sketch both graphs and read off where one lies above the other. The first often turns the inequality into a sign question about a single quantity such as $\cos 2x$.

Strategy: move to one side, factorise, read the signs

Write " $f(x) > g(x)$ " as " $f - g > 0$ ", factorise using identities, and read the sign of each factor on the given interval (a factor like $\sin x$ or $\cos x$ is often *known to be positive* on $(0, \frac{\pi}{2})$, so you can drop it safely). Only divide by a quantity whose sign you know.

Worked example: which is larger, $\tan x$ or $\sin 2x$?

Question. For $0 < x < \frac{\pi}{2}$, find the values of x for which $\tan x > \sin 2x$. What fraction of the interval $0 \leq x < 2\pi$ satisfies $\sin 2x > \cos 2x$?

Solution. On $(0, \frac{\pi}{2})$, $\sin x > 0$ and $\cos x > 0$. Now $\tan x - \sin 2x = \frac{\sin x}{\cos x} - 2 \sin x \cos x = \frac{\sin x}{\cos x} (1 - 2 \cos^2 x) = -\frac{\sin x}{\cos x} \cos 2x$. The fraction $\frac{\sin x}{\cos x}$ is positive, so the difference is positive exactly when $\cos 2x < 0$, i.e. $2x \in (\frac{\pi}{2}, \pi)$, which means $\frac{\pi}{4} < x < \frac{\pi}{2}$.

For the fraction: $\sin 2x > \cos 2x$ is the same as $\sin u > \cos u$ with $u = 2x$, and on every full period of u the graphs of $\sin u$ and $\cos u$ cross twice and each is above the other for exactly half of the period (the curves are the same shape, shifted). The interval $0 \leq x < 2\pi$ gives $u \in [0, 4\pi)$, two full periods, so the fraction is $\frac{1}{2}$.

Examiner's eye. Check a value: $x = \frac{\pi}{3}$ gives $\tan = \sqrt{3} \approx 1.73$ and $\sin 2x = \frac{\sqrt{3}}{2} \approx 0.87$, so $\tan x > \sin 2x$ and $\frac{\pi}{3} > \frac{\pi}{4}$. And at $x = \frac{\pi}{6}$: $\tan = \frac{1}{\sqrt{3}} \approx 0.58 < \sin \frac{\pi}{3} \approx 0.87$. The boundary is $\frac{\pi}{4}$, where both equal 1.

Habits that win marks

- **Sketch the graph** before you start whenever the question says “how many”, “for which x ” or compares two functions.
- **Name the quadrant** and write the sign before you write a value.
- **Reduce to one function of one angle**, then substitute t and check $-1 \leq t \leq 1$.
- **Factorise instead of dividing**; check end points of the interval separately.
- **Multiple angles multiply the count** (for $\sin kx = c$ on a full period of x), but half-periods do not.
- **Test a value or a special angle** ($\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$) in the final answer. Exact values make arithmetic easy: use them to verify.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Find the exact value of $\cos \frac{7\pi}{6}$.
2. Find the exact value of $\sin^2 15^\circ + \sin^2 75^\circ$.
3. How many solutions has $\cos 2x = \cos x$ for $0 \leq x \leq 2\pi$?
4. How many solutions has $\tan x = 2 \sin x$ for $0 \leq x < 2\pi$?
5. In triangle ABC , $A = 45^\circ$, $B = 60^\circ$ and $a = \sqrt{2}$. Find b .
6. Two sides of a triangle are 6 and 8 with an angle of 30° between them. Find its area.
7. How many solutions has $\sin 3x = \frac{1}{4}$ for $0 \leq x \leq \pi$?
8. A sector has radius 6 and angle $\frac{2\pi}{3}$ radians. Find its area and arc length.

Answers to the self-check

1. $\frac{7\pi}{6} = \pi + \frac{\pi}{6}$ is in the third quadrant, so $\cos$ is negative: $-\frac{\sqrt{3}}{2}$.
2. $\sin 75^\circ = \cos 15^\circ$, so the sum is $\sin^2 15^\circ + \cos^2 15^\circ = 1$.
3. $2\cos^2 x - \cos x - 1 = 0$, i.e. $(2\cos x + 1)(\cos x - 1) = 0$. $\cos x = -\frac{1}{2}$ gives $\frac{2\pi}{3}, \frac{4\pi}{3}$; $\cos x = 1$ gives 0 and 2π (both are in the closed interval). Total 4.
4. $\sin x = 2\sin x \cos x$ (with $\cos x \neq 0$), so $\sin x(1 - 2\cos x) = 0$: $\sin x = 0$ gives $0, \pi$; $\cos x = \frac{1}{2}$ gives $\frac{\pi}{3}, \frac{5\pi}{3}$. Total 4. ($\frac{\pi}{2}$ and $\frac{3\pi}{2}$ are not solutions: $\tan$ is undefined there.)
5. $b = \frac{a \sin B}{\sin A} = \frac{\sqrt{2} \cdot \frac{\sqrt{3}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{3}$.
6. $\frac{1}{2} \cdot 6 \cdot 8 \cdot \sin 30^\circ = 12$.
7. $3x$ runs over $[0, 3\pi]$. The sine is positive on $(0, \pi)$ and on $(2\pi, 3\pi)$, and each positive arch crosses $\frac{1}{4}$ twice, while $(\pi, 2\pi)$ gives none. Total 4.
8. Area $\frac{1}{2} \cdot 36 \cdot \frac{2\pi}{3} = 12\pi$; arc length $6 \cdot \frac{2\pi}{3} = 4\pi$.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.