

TMUA

Paper 2

Necessary and sufficient conditions

Free lesson notes

Covers	2.1 if-then, iff, converse, contrapositive, necessary, sufficient
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Roughly one Paper 2 question in four hides a logic word (*necessary, sufficient, only if, converse, contrapositive*) inside a small piece of ordinary mathematics. The mathematics is easy: a quadratic, a remainder, an integral of an odd function. The trap is *direction*. Students who know the algebra still lose the mark because they decide that “ P is necessary for Q ” means the wrong way round.

This lesson fixes the meaning of every phrase once, turns it into a picture (one set inside another), and then builds a fast routine for the exact option formats the TMUA uses: “which of I, II, III is sufficient”, the eight-row yes/no table, and the four-way “necessary and/or sufficient” choice. There are ten worked examples, each tied to a recurring pattern, and each answer was verified by computer as well as by hand.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- “Which of I, II, III is/are sufficient (or necessary)” with the options A none of them, B I only, ..., H I, II and III. Decide each condition on its own, then read off the letter.
- **The yes/no table:** eight rows, three columns (“Condition I is sufficient: yes/no” and so on). Same task, different packaging.
- **Four-way classification:** is P necessary and sufficient, necessary but not sufficient, sufficient but not necessary, or neither?
- **Equivalent statements:** given an “if... then” sentence (often with “only if”), which of several sentences say the same thing: the contrapositive, but never the converse.
- **Conditions on functions:** odd, even, constant, or $f(a - x) = f(a + x)$, linked to integrals over symmetric intervals or to symmetry of a graph.
- **Geometry:** which condition on a parallelogram makes it a rectangle, a rhombus, a square.
- **Number conditions:** remainders on division by 6 and a trigonometric value, divisibility, and statements about the mean and median of a list or the number of roots of an equation.

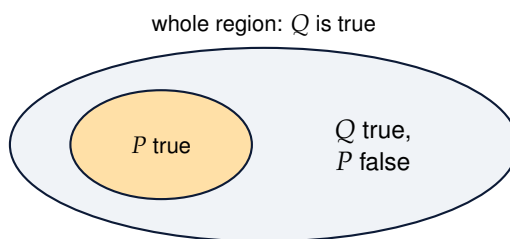
What if, only if and iff really say

Everything on Paper 2 about conditions is built on one idea. “ $P \Rightarrow Q$ ” means: *every time P is true, Q is true as well*. There are many ways to say it, and the TMUA uses all of them.

One implication, six costumes

If P then Q	/	P implies Q	$P \Rightarrow Q$		
Q if P	/	Q whenever P	/	Q provided P	$P \Rightarrow Q$
P only if Q			$P \Rightarrow Q$		
P is sufficient for Q			$P \Rightarrow Q$		
Q is necessary for P			$P \Rightarrow Q$		
P if and only if Q	/	P is necessary and sufficient for Q	$P \Rightarrow Q$ and $Q \Rightarrow P$		

The cleanest way to remember it is a picture. If $P \Rightarrow Q$, the region where P is true sits *inside* the region where Q is true:



Every point where P is true lies inside the region where Q is true, so $P \Rightarrow Q$. The small set P is the *strong* (sufficient) condition; the big set Q is the *weak* (necessary) one.

From $P \Rightarrow Q$ you can form three relatives. The **converse** is $Q \Rightarrow P$. The **inverse** is $\neg P \Rightarrow \neg Q$. The **contrapositive** is $\neg Q \Rightarrow \neg P$ ("if Q fails then P fails"). In the picture, the contrapositive says "outside Q means outside P ", which is obviously the same fact. So:

Equivalence in one line

The contrapositive is **always equivalent** to the original (both true or both false). The converse and the inverse are equivalent to each other, but **not** to the original. A true statement can have a false converse.

Worked example: which sentences say the same thing?

Question. A positive integer n is a multiple of 15 **only if** n is a multiple of 5. Which of the following say the same thing?

- I If n is a multiple of 15 then n is a multiple of 5.
- II If n is a multiple of 5 then n is a multiple of 15.
- III If n is not a multiple of 5 then n is not a multiple of 15.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. Let P : " $15 \mid n$ " and Q : " $5 \mid n$ ". " P only if Q " is $P \Rightarrow Q$. Statement I is exactly $P \Rightarrow Q$. Statement II is $Q \Rightarrow P$, the converse (and it is false: $n = 5$). Statement III is $\neg Q \Rightarrow \neg P$, the contrapositive. So I and III say the same thing and II does not: **F**.

Examiner's eye. Never translate by feel. Name P and Q first, write each option as an arrow, and compare arrows. "Only if" hides the arrow pointing *towards* the word after it.

Worked example: proving an if-and-only-if

Question. Prove that $n(n+1)$ is divisible by 4 if and only if n leaves remainder 0 or 3 on division by 4.

Solution. An "iff" needs two proofs.

($\Leftarrow$) If $n = 4k$ then $n(n+1) = 4k(4k+1)$ is a multiple of 4. If $n = 4k+3$ then $n(n+1) = (4k+3)(4k+4) = 4(4k+3)(k+1)$, a multiple of 4.

($\Rightarrow$) Use the contrapositive: if n leaves remainder 1 or 2 then $n(n+1)$ is *not* divisible by 4. For $n = 4k+1$: $n(n+1) = (4k+1)(4k+2) = 2(4k+1)(2k+1)$, which is $2 \times$ an odd number. For $n = 4k+2$: $(4k+2)(4k+3) = 2(2k+1)(4k+3)$, again $2 \times$ an odd number. A number of the form $2 \times$ odd is not a multiple of 4.

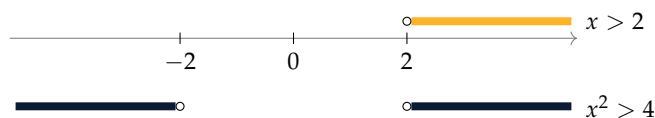
Examiner's eye. The contrapositive was the easy route for ($\Rightarrow$): it is simpler to start from the remainders and show failure than to start from divisibility and extract the remainders. When " $P \Rightarrow Q$ " is awkward, try $\neg Q \Rightarrow \neg P$.

Necessary and sufficient: sets, and two one-way tests

Take a condition P and a target statement Q (the thing you are being asked about). Then:

- P is **sufficient** for Q if $P \Rightarrow Q$: the P -region is inside the Q -region.
- P is **necessary** for Q if $Q \Rightarrow P$: the Q -region is inside the P -region.

Look at the number line for $x > 2$ (the condition) and $x^2 > 4$ (the target):



Everything with $x > 2$ also has $x^2 > 4$, so $x > 2$ is sufficient. But $x = -3$ has $x^2 > 4$ and not $x > 2$, so $x > 2$ is not necessary.

Strategy: two one-way tests

Test the two directions *separately*, and use the cheap tool for each. To **refute sufficiency**, find one example where P is true and Q is false. To **refute necessity**, find one example where Q is true and P is false. If your hunt for such an example fails, stop guessing and **prove** the implication (an example can never prove a “for all”). Always write “ $P \Rightarrow Q$?” and “ $Q \Rightarrow P$?” on your paper before looking at the options.

Worked example: four ways to be a condition

Question. For a positive integer n , let Q be: “ n^2 is divisible by 12”. Classify each of the following conditions on n using the options

- A necessary and sufficient B necessary but not sufficient
C sufficient but not necessary D neither necessary nor sufficient

(i) n is divisible by 6. (ii) n is divisible by 12. (iii) n is even. (iv) n is divisible by 4.

Solution. First understand Q . Since $12 = 4 \times 3$ with 4 and 3 coprime, $12 \mid n^2$ means $4 \mid n^2$ and $3 \mid n^2$. A square is divisible by 4 exactly when the number is even, and by 3 exactly when the number is a multiple of 3. So $Q \iff 6 \mid n$.

(i) divisible by 6: equivalent to Q , so **A**. (ii) divisible by 12 implies divisible by 6 (sufficient) but $n = 6$ has $36 = 3 \times 12$ and is not a multiple of 12: **C**. (iii) even: necessary, since Q forces n to be even, but $n = 2$ gives 4: not sufficient: **B**. (iv) divisible by 4: $n = 6$ satisfies Q but not (iv), so not necessary; $n = 4$ has 16 not divisible by 12, so not sufficient: **D**.

Examiner’s eye. Simplifying Q first (to “ $6 \mid n$ ”) made all four verdicts one-line checks. Rewrite the target in its simplest equivalent form before you compare.

Worked example: which are necessary?

Question. For a real number x the inequality $(x - 1)(x - 4) < 0$ holds. Which of the following is/are *necessary* for it? I: $x > 0$. II: $x^2 < 16$. III: $x < 3$.

- A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. The inequality holds exactly for $1 < x < 4$. A necessary condition must be true for *every* such x . I: $x > 0$ is true throughout $(1, 4)$: necessary. II: $x^2 < 16$ means $-4 < x < 4$, which contains $(1, 4)$: necessary. III: $x < 3$ fails at $x = 3.5$, which satisfies the inequality: not necessary. So I and II only: **E**.

Examiner’s eye. Necessary conditions are *weaker* than the target: they are allowed to be true in extra places. A necessary condition is a bigger window (here $x > 0$ and $-4 < x < 4$), a sufficient one a smaller window.

Common mistakes

- Reading “ P is necessary for Q ” as $P \Rightarrow Q$. It is $Q \Rightarrow P$: the thing you are asked *about* (P) is the weaker, wider one.
- Using many examples where both hold as “evidence”. Only an example with one true and one false decides anything.
- Forgetting strictness: $x > 2$ is sufficient for $x \geq 2$ but not necessary; $x \geq 2$ is necessary for $x > 2$ but not sufficient.

Conditions on functions and shapes

Two function facts do most of the work. A function is **even** if $f(-x) = f(x)$ (symmetric about the y -axis) and **odd** if $f(-x) = -f(x)$ (rotationally symmetric about the origin). Over a symmetric interval,

$$\int_{-a}^a f(x) dx = 0 \text{ if } f \text{ is odd,} \quad \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx \text{ if } f \text{ is even.}$$

The graph of $y = f(x)$ is symmetric about the line $x = a$ exactly when $f(a + t) = f(a - t)$ for every t : points at the same distance either side of $x = a$ have equal heights. Equivalent: $f(x) = f(2a - x)$, where $2a - x$ is the mirror image of x in a .

Worked example: an integral over a symmetric interval

Question. f is continuous on $[-1, 1]$ and Q is the statement $\int_{-1}^1 f(x) dx = 0$. Which of the following is/are *sufficient* for Q ? I: f is an odd function. II: $f(x) = 3x^2 - 1$ for all x in $[-1, 1]$. III: f is an even function.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. I: odd functions integrate to 0 over $[-a, a]$: sufficient. II: $\int_{-1}^1 (3x^2 - 1) dx = [x^3 - x]_{-1}^1 = 0 - 0 = 0$: sufficient (it is even, so being even cannot be what matters). III: $f(x) = 1$ is even with integral $2 \neq 0$: not sufficient. So I and II only: E. Neither I nor II is *necessary*: $f(x) = x^2 - \frac{1}{3}$ is not odd, is not $3x^2 - 1$, and has integral $\frac{2}{3} - \frac{2}{3} = 0$.

Examiner's eye. A "condition" can be a concrete formula (like II). It is sufficient if every function satisfying it makes Q true, and you check that with the one formula.

Worked example: symmetry of a graph

Question. The graph of $y = g(x)$ is symmetric about the line $x = 2$. Which of the following must be true? I: $g(5) = g(-1)$. II: $g(0) = g(2)$. III: $g(1 + t) = g(3 - t)$ for every real t .

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. The mirror image of x in the line $x = 2$ is $4 - x$, and symmetry says $g(x) = g(4 - x)$. I: the mirror image of 5 is -1 : true. II: the mirror image of 0 is 4, not 2, so $g(0) = g(4)$; nothing forces $g(0) = g(2)$ (try $g(x) = (x - 2)^2$: $4 \neq 0$): not necessary. III: the mirror image of $1 + t$ is $4 - (1 + t) = 3 - t$: true. So I and III only: F.

Examiner's eye. The one-line method: write the mirror map $x \mapsto 2a - x$ and apply it to each claimed pair. A claim is forced exactly when the two inputs are mirror images.

Worked example: parallelograms

Question. $ABCD$ is a parallelogram. Which of the following is/are *sufficient* for $ABCD$ to be a rectangle? I: the diagonals AC and BD have equal length. II: angle ABC is a right angle. III: the diagonals are perpendicular.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. Let $\mathbf{u} = \overrightarrow{AB}$ and $\mathbf{v} = \overrightarrow{AD}$. The diagonals are $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$. I: $|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u} - \mathbf{v}|^2$ gives $4\mathbf{u} \cdot \mathbf{v} = 0$, so the sides are perpendicular: a rectangle. Sufficient. II: a parallelogram with one right angle has all four (opposite angles equal, adjacent angles sum to 180°): sufficient. III: perpendicular diagonals means $|\mathbf{u}| = |\mathbf{v}|$, a rhombus. A rhombus that is not a square (for example $\mathbf{u} = (2, 1)$, $\mathbf{v} = (1, 2)$) is not a rectangle. So I and II only: E.

Examiner's eye. Hierarchy to remember: square $\subset$ rectangle and rhombus $\subset$ parallelogram. A condition is sufficient for a rectangle only if it forces a right angle; perpendicular diagonals force equal sides instead.

Numbers, lists and solutions

For number statements, translate every condition into the same language (usually remainders) and compare lists of allowed values. For statements that "must be true", hunt for a single example where the claim fails: one failure removes it.

Worked example: remainders and a trig value

Question. For an integer n , let Q be the statement $\cos\left(\frac{n\pi}{3}\right) = \frac{1}{2}$. Consider I: $n \equiv 1 \pmod{6}$. II: n is odd. III: n is odd and not a multiple of 3. Which of them are **sufficient** for Q ? Which are **necessary**? Answer each with the yes/no table.

	I sufficient?	II sufficient?	III sufficient?
A	yes	yes	yes
B	yes	yes	no
C	yes	no	yes
D	yes	no	no
E	no	yes	yes
F	no	yes	no
G	no	no	yes
H	no	no	no

Solution. The cosine repeats every 6 steps of $\pi/3$: $\cos(n\pi/3)$ for $n \equiv 0, 1, 2, 3, 4, 5 \pmod{6}$ is $1, \frac{1}{2}, -\frac{1}{2}, -1, -\frac{1}{2}, \frac{1}{2}$. So Q holds exactly when $n \equiv 1$ or $5 \pmod{6}$.

Sufficient? I: $n \equiv 1$ gives $\frac{1}{2}$: yes. II: $n = 3$ is odd but $\cos \pi = -1$: no. III: odd and not a multiple of 3 means $n \equiv 1, 5 \pmod{6}$: yes. The row is yes, no, yes: **C**.

Necessary? I: $n = 5$ satisfies Q but $5 \not\equiv 1$: no. II: $n \equiv 1, 5$ is always odd: yes. III: same list as Q : yes. Reading the table for “necessary” (no, yes, yes): the row is **E**.

Examiner’s eye. Both questions use the same three conditions, so the real work is the one-off translation of Q into “ $n \equiv \pm 1 \pmod{6}$ ”. After that, each verdict is a comparison of two lists.

Worked example: mean and median

Question. Five integers $a \leq b \leq c \leq d \leq e$ are written in increasing order. Let Q be: “the mean equals the median”. Classify each condition with the four-way options

A necessary and sufficient B necessary but not sufficient
C sufficient but not necessary D neither necessary nor sufficient

(i) a, b, c, d, e are in arithmetic progression. (ii) $a + b + d + e = 4c$. (iii) $a + b + c + d + e$ is divisible by 5. (iv) one of the numbers is 3.

Solution. The median is c and the mean is $(a + b + c + d + e)/5$, so $Q \iff a + b + c + d + e = 5c \iff a + b + d + e = 4c$. Hence (ii) is necessary and sufficient: **A**. (i) An arithmetic progression has $a + e = b + d = 2c$, so $a + b + d + e = 4c$: sufficient; but $0, 1, 3, 5, 6$ has mean 3 and median 3 without being a progression: **C**. (iii) If the mean equals c then the sum is $5c$, a multiple of 5: necessary; but $0, 0, 0, 0, 5$ has sum 5, mean 1, median 0: **B**. (iv) $0, 0, 0, 0, 0$ satisfies Q with no 3 (not necessary), and $0, 0, 3, 3, 5$ has median 3, mean 2.2 (not sufficient): **D**.

Examiner’s eye. The statement “mean = median” looks statistical but is one linear equation. Turning it into algebra decides all four parts at once.

Worked example: what must follow

Question. The equation $f(x) = 0$ has exactly three distinct real roots. Which of the following must be true? I: $f(x) + 1 = 0$ has exactly three distinct real roots. II: $f(x + 4) = 0$ has exactly three distinct real roots. III: $f(-x) = 0$ has exactly three distinct real roots.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. Shifting or reflecting the graph sideways moves or mirrors the roots but never changes how many there are. II: the roots of $f(x + 4) = 0$ are each root of f minus 4: still three. III: the roots of $f(-x) = 0$ are the negatives of the roots of f : still three. I moves the graph *vertically*, and then the number of crossings with the axis can change: for $f(x) = x^3 - x$ (roots $-1, 0, 1$), $f(x) + 1 = x^3 - x + 1$ has only one real root. So II and III only: **G**.

Examiner’s eye. A horizontal change ($x \mapsto x + 4$, $x \mapsto -x$, $x \mapsto 2x$) rearranges the roots. A vertical change ($+1$) can create or destroy roots. That one distinction answers a whole family of questions.

Habits that win marks

- Name P and Q before you read the options, and write both $P \Rightarrow Q$? and $Q \Rightarrow P$? on your page.
- Remember the picture: sufficient is the small set, necessary is the big set. " P is necessary for Q " means Q lives inside P .
- Decide each of I, II, III independently, write yes/no beside each, then convert to the option letter. Never choose the letter first.
- Refute with one example (one true, one false). If you cannot find one, prove the implication in general.
- Simplify the target statement to its cleanest form (a remainder condition, an interval, a single equation) before comparing.
- The contrapositive is equivalent to the original; the converse is not. When a statement is hard to prove, try its contrapositive.
- Test strictness at the end points: $x > 2$ against $x \geq 2$ changes the answer.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. For real x , is " $x^2 = x$ " necessary, sufficient, both or neither for " $x = 1$ "?
2. Write the contrapositive of "if ab is odd then a is odd" and say whether the converse is true.
3. "A triangle is equilateral only if it is isosceles." Write this as an arrow and say whether it is true.
4. For integers n , classify " n is a multiple of 4" as a condition for " n^2 is divisible by 8".
5. Is " $x^2 - 5x + 6 = 0$ " necessary, sufficient, both or neither for " $x = 2$ "?
6. Which is equivalent to "if a quadrilateral is a square then its diagonals are equal": (a) if the diagonals are equal then it is a square; (b) if the diagonals are not equal then it is not a square?
7. Is " n is a multiple of 3" necessary, sufficient, both or neither for " n is a multiple of 6"?
8. Is " f is even" necessary for $\int_{-1}^1 f(x) dx = 2 \int_0^1 f(x) dx$?

Answers to the self-check

1. $x = 1 \Rightarrow x^2 = x$, so it is necessary. But $x = 0$ also satisfies $x^2 = x$, so it is not sufficient. Necessary but not sufficient.
2. Contrapositive: if a is even then ab is even. The converse (if a is odd then ab is odd) is false: $a = 1, b = 2$.
3. equilateral $\Rightarrow$ isosceles. True. (The converse is false.)
4. If $n = 4k$ then $n^2 = 16k^2$ is divisible by 8. Conversely if $8 \mid n^2$ then n is even, $n = 2m$, so $8 \mid 4m^2$ forces m even, so $4 \mid n$. Necessary and sufficient.
5. $x = 2$ gives $4 - 10 + 6 = 0$, so it is necessary. $x = 3$ also satisfies it, so it is not sufficient. Necessary but not sufficient.
6. (b), the contrapositive. (a) is the converse and is false (any rectangle that is not a square).
7. $6 \mid n \Rightarrow 3 \mid n$, so necessary. $n = 3$ shows it is not sufficient. Necessary but not sufficient.
8. No. $f(x) = x^3 - \frac{x}{2}$ is odd, not even, yet $\int_{-1}^1 f = 0$ and $2 \int_0^1 f = 2(\frac{1}{4} - \frac{1}{4}) = 0$. (Even *is* sufficient, so here it is sufficient but not necessary.)

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.