

TMUA

Paper 2

Negation and quantifiers

Free lesson notes

Covers	2.2 for all, there exists, negating statements, order of quantifiers
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Negating a sentence correctly is one of the most tested skills on Paper 2, and it is the one where English misleads you. “Not all students passed” does not mean “no student passed”. “For every m there is an n ” is a completely different claim from “there is an n for every m ”. A single quantifier in the wrong place changes the truth of a statement, and the TMUA exploits this in eight-option questions where every wrong answer is a plausible slip.

This lesson gives you a mechanical, error-proof procedure (flip each quantifier from left to right, then flip the inner claim), shows why *order* matters with a simple game, and then practises the exact formats: choosing the negation from eight candidates, deciding which of I, II, III is true, and finding the value of k that makes a “for all...if...then” sentence true. There are nine worked examples, each answer verified by brute force.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **Choose the negation from eight candidates:** a sentence with two quantifiers and an inner “and”/“or”, and seven slips among the options (a quantifier not flipped, an inner connective not flipped, order swapped).
- **Which of I, II, III is true?:** statements with “for all...there exists...” over integers or reals, where reversing the order changes the truth.
- **A parameter question:** “for all real x , if $x < k$ then $x^2 < k$ ” is true for which values of k ? You are really asked to find when the negation has no witness.
- **Statements about primes, integers and sequences:** “every prime...”, “there is exactly one...”, “for every n there is $m > n$ with $u_m > u_n$ ”.
- **Vacuous truth:** an “if...then” whose “if” can never happen is true, and the TMUA likes to test whether you know it.
- **Negating an “if...then”:** the negation is “ P and not Q ”, which is exactly what a counterexample shows.

The flip rules, and a method that never fails

A statement is true or false; its negation is true exactly when the statement is false. The rules are short enough to learn by heart:

Negation rules

Statement	Negation
for all x , $P(x)$	there exists x such that not $P(x)$
there exists x such that $P(x)$	for all x , not $P(x)$
P and Q	(not P) or (not Q)
P or Q	(not P) and (not Q)
if P then Q	P and (not Q)
there is exactly one x with $P(x)$	no x has $P(x)$, or at least two do

Inequalities flip as expected: $\neg(x > 5)$ is $x \leq 5$ (not $x < 5$), and $\neg(a < b < c)$ is $a \geq b$ or $b \geq c$. “Or” is always inclusive:

“ P or Q ” is true when both are true.

Strategy: flip left to right

Read the sentence from the left. Every time you meet a quantifier, swap it ($\forall \leftrightarrow \exists$) and keep its position. When you reach the inner claim, negate that ($\wedge \leftrightarrow \vee$, $\leq \leftrightarrow >$, and so on). Do *not* reorder anything, and do not change the variables. Write your negation *before* you look at the options, then match it exactly.

Worked example: one of eight

Question. Consider the statement

“For every integer m there is an integer n such that $n > m$ and n is a multiple of 5.”

Which of the following is the negation of this statement?

- A For every integer m there is an integer n such that $n \leq m$ or n is not a multiple of 5.
- B There is an integer m such that every integer n has $n \leq m$ and is not a multiple of 5.
- C There is an integer n such that every integer m has $n \leq m$ or n not a multiple of 5.
- D There is an integer m such that every integer n has $n \leq m$ or is not a multiple of 5.
- E Every integer m and every integer n satisfy $n \leq m$ or n is not a multiple of 5.
- F For every integer m there is an integer n with $n > m$ and n not a multiple of 5.
- G There are integers m and n with $n \leq m$ or n not a multiple of 5.
- H There is an integer m such that every integer n has $n > m$ and is a multiple of 5.

Solution. Flip left to right. “For every m ” becomes “there is an m such that”. “There is an n ” becomes “for every n ”. The inner claim “ $n > m$ and n is a multiple of 5” becomes “ $n \leq m$ or n is not a multiple of 5”. Result: “There is an integer m such that every integer n has $n \leq m$ or is not a multiple of 5.” That is **D**.

Why the others fail: A keeps $\forall \exists$; B negates “and” as “and”; C swaps the order of m and n ; E turns the $\exists$ into a $\forall$; F negates only half of the inner claim; G has $\exists \exists$; H does not negate the inner claim at all. (The original statement is true: take the next multiple of 5 above m . So its negation is false.)

Examiner’s eye. The check is mechanical: if you have flipped every quantifier and the connective, you hold exactly one option. Options that differ from yours in a single place are the traps.

Worked example: not all is not none

Question. Every student in a class likes mathematics or physics (or both). Which of the following is the negation of this statement? I: some student likes neither mathematics nor physics. II: every student dislikes both mathematics and physics. III: some student does not like mathematics or does not like physics.

- A none of them B I only C II only D III only
- E I and II only F I and III only G II and III only H I, II and III

Solution. The statement is “for all students, M or P ”. Its negation is “there is a student with not M and not P ”: someone who likes neither. That is I. II is too strong: take a class in which one student likes neither subject and the others like maths. The original statement is false there (so its negation is true), yet II is false. III is too weak: a class in which every student likes maths but not physics makes III true, while the original statement is true there (so its negation is false). So I only: **B**.

Examiner’s eye. “Not all” means “at least one fails”. Whenever an option starts with “every” or “no”, ask whether the negation really needs a statement about *all* cases. It almost never does.

The order of quantifiers

Think of a quantified statement as a game. For $\forall x \exists y \dots$ an opponent chooses x first (any value, to make life hard) and then you choose y , knowing x . For $\exists y \forall x \dots$ you must choose y first, and the opponent then picks any x to defeat you. Choosing second is much easier, so $\forall x \exists y$ is a weaker claim than $\exists y \forall x$. The second always implies the first, but not the other way round.

A standard contrast: over the integers, “for every x there is a y with $x + y = 0$ ” is true ($y = -x$), but “there is a y such that for every x , $x + y = 0$ ” is false (the single y cannot cancel every x).

Worked example: which are true?

Question. Which of the following statements is/are true? I: for every real x there is a real y with $y^2 = x$. II: for every integer n there is an integer m with $m > n^2$. III: there is a real y such that $y < x$ for every positive real x .

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. I: false. The opponent picks $x = -1$ and no real y has $y^2 = -1$. II: true. Whatever n is chosen, $m = n^2 + 1$ works. (Compare “there is an integer m greater than n^2 for every n ”, which is false because n^2 has no upper bound.) III: true. You may choose y first, and $y = 0$ is below every positive x . So II and III only: **G**.

Examiner's eye. For III the order is “ $\exists y$ first”, which looks harder, yet it is true because the statement only asks for one fixed y that works against every positive x at once. Read the order, then play the game.

Worked example: sequences and quantifiers

Question. The sequence $u_n = (n - 5)^2$ is defined for $n = 1, 2, 3, \dots$. Which of these is/are true? I: for every n there is $m > n$ with $u_m > u_n$. II: there is an n such that $u_m > u_n$ for every $m > n$. III: for every n there is $m > n$ with $u_m < u_n$.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. The terms are 16, 9, 4, 1, 0, 1, 4, 9, ...: they fall to 0 at $n = 5$ and then increase for ever. I: true, since the sequence grows without bound, so a later bigger term always exists. II: true with $n = 5$ (or any larger n): every later term exceeds $u_5 = 0$. III: false. Take $n = 6$: $u_6 = 1$ and every later term is larger, so no $m > 6$ has $u_m < u_6$. So I and II only: **E**.

Examiner's eye. To disprove a “for every n there is m ” you need a *single bad* n (here $n = 6$) for which *no* later m works. To prove a “there is an n ” you only need one good n .

If-then, a parameter, and “exactly one”

The negation of “if P then Q ” is “ P and not Q ”. This is why a counterexample to a universal “if... then” claim must *satisfy the hypothesis and break the conclusion*. A statement “for all x , if $P(x)$ then $Q(x)$ ” is true exactly when no x makes P true and Q false.

Worked example: find the values of k

Question. For which real values of k is the following statement true? “For every real number x , if $x > k$ then $x^2 > k$.”

Solution. Work with the negation: “there is a real x with $x > k$ and $x^2 \leq k$ ”. The statement is true exactly when this has *no* solution.

If $k < 0$: $x^2 \leq k < 0$ is impossible, so no witness; the statement is true. If $k = 0$: $x^2 \leq 0$ forces $x = 0$, but we need $x > 0$; no witness; true. If $0 < k < 1$: take $x = \sqrt{k}$. Then $x > k$ (because $\sqrt{k} > k$ when $0 < k < 1$) and $x^2 = k \leq k$; a witness exists, so the statement is false. If $k \geq 1$: any $x > k$ has $x^2 > x > k$, so no witness; true.

The statement is true for $k \leq 0$ and for $k \geq 1$ (false exactly for $0 < k < 1$).

Examiner's eye. The answer is a *gap* in the middle ($0 < k < 1$), not a single threshold. Test a value inside the gap (say $k = \frac{1}{4}$: $x = \frac{1}{2}$ gives $x > k$ but $x^2 = k$) and one outside ($k = 2$: $x > 2$ gives $x^2 > 4 > 2$).

Worked example: negating a claim about primes

Question. Write down the negation of “for every prime p , the number $2p + 1$ is prime”, and decide which of the two is true.

Solution. Negation: “there is a prime p such that $2p + 1$ is not prime”. Try small primes: $p = 2 \rightarrow 5$, $p = 3 \rightarrow 7$, $p = 5 \rightarrow 11$ are prime, so the pattern looks promising, but $p = 7$ gives $15 = 3 \times 5$. So the negation is true and the original statement is false. A single counterexample ($p = 7$) settles it; the three successes before it prove nothing.

Examiner's eye. This is the bridge to the next lesson: the negation of a universal claim is an existence claim, and an existence claim is proved by producing one example.

Worked example: exactly one

Question. “There is exactly one prime p such that $p + 2$ and $p + 4$ are also prime.” Is this true? Describe its negation.

Solution. True. $p = 3$ works (3, 5, 7). For a prime $p > 3$ look at $p, p + 2, p + 4$ modulo 3: they leave three different remainders $r, r + 2, r + 1$ (all of 0, 1, 2), so one of them is a multiple of 3 and larger than 3, hence not prime. So 3 is the only one.

The negation of “exactly one” is “**none, or at least two**”: either no prime has the property, or there are two different primes with it. (In this case neither happens.)

Examiner’s eye. “Exactly one” hides two claims, existence and uniqueness, so its negation is an “or” of two failures. A common slip is to negate it as “exactly two” or “at most one”.

Vacuous truth, and negating a property

An “if... then” statement with a hypothesis that can never be satisfied is **true**. Why? Its negation is “ P and not Q ”, and if P never happens that can never be true, so the original cannot be false. Examples: “every integer n with $n^2 < 0$ is odd” (there are none) or “all the elephants in this room are pink” when the room has none.

Worked example: vacuously true

Question. Which of these is/are true? I: for every integer n , if n is a multiple of 4 and n is odd then $n = 17$. II: there is a prime p such that $p + 1$ and $p + 2$ are both prime. III: for every real x , if $x^2 < 0$ then $x^2 > 0$.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. I: no integer is both a multiple of 4 and odd, so the hypothesis never holds and the statement is true (vacuously), even though the conclusion looks silly. II: false. If p is odd then $p + 1$ is even and bigger than 2, so not prime; if $p = 2$ then $p + 2 = 4$ is not prime. III: true for the same reason as I: no real x has $x^2 < 0$. So I and III only: F.

Examiner’s eye. When you meet an “if... then” about all objects, ask first: *can the hypothesis ever be true?* If not, the whole statement is true and you need nothing else.

Worked example: negating “is increasing”

Question. A function f is strictly increasing on $\mathbb{R}$ if for all real a and b , if $a < b$ then $f(a) < f(b)$. Which of the following functions is/are **not** strictly increasing? I: $f(x) = x^3$. II: $f(x) = x^3 - x$. III: $f(x) = x|x|$.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. The negation of “increasing” is “there exist $a < b$ with $f(a) \geq f(b)$ ”. So we hunt for one such pair. I: x^3 never loses ground: no pair, so it is increasing. II: $a = -1, b = 0$ gives $f(-1) = 0 \geq 0 = f(0)$: a pair exists, so $x^3 - x$ is not increasing. III: $x|x|$ equals x^2 for $x \geq 0$ and $-x^2$ for $x < 0$; it rises through the origin and has no bad pair. So only II: C.

Examiner’s eye. “Not increasing” does *not* mean “decreasing”. $x^3 - x$ rises, falls and rises again: it is neither increasing nor decreasing. The negation only asks for one bad pair.

Habits that win marks

- Write your own negation first, using the flip-left-to-right method, and only then read the options. Match exactly.
- The order of quantifiers is part of the meaning. Never reorder them, and read $\forall \exists$ as “opponent chooses first, I answer”.
- Negating “and” gives “or”, negating “or” gives “and”, negating $\leq$ gives $>$ (never $\geq$).
- The negation of “if P then Q ” is “ P and not Q ”, never “if P then not Q ”.
- “Not all” means “at least one fails”, not “none”. “Not exactly one” means “none, or at least two”.
- To disprove a “for all” claim, find one witness to the negation. To prove a “there exists”, exhibit one example.
- An “if... then” with an impossible hypothesis is true. Test whether the hypothesis can ever hold.
- For a parameter question, solve for when the negation has *no* solution, and check each boundary value separately.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Write the negation of "some square numbers are even".
2. Negate: "for all real x , $x^2 \geq x$ ". Which of the two is true?
3. Negate: " $n > 3$ or n is even".
4. Negate: "for every integer m there is an integer n with $n^2 > m$ ". Which is true?
5. Is "there is a real y such that $y^2 > x$ for every real x " true?
6. Negate: "if ab is even then a is even or b is even". Is the original true?
7. Negate: "there is a number M with $u_n \leq M$ for all n " (the sequence is bounded above).
8. Is "every real number x with $x^2 = -1$ is greater than 5" true?

Answers to the self-check

1. "Every square number is odd" ("no square number is even").
2. "There is a real x with $x^2 < x$ ": true, for example $x = \frac{1}{2}$. So the original is false.
3. " $n \leq 3$ and n is odd."
4. "There is an integer m such that for every integer n , $n^2 \leq m$." False (take $n = |m| + 1$), so the original is true.
5. No: for any y , taking $x = y^2$ gives $y^2 > y^2$, which is false.
6. " ab is even, a is odd and b is odd." This is impossible (odd $\times$ odd is odd), so the negation is false and the original is true.
7. "For every M there is an n with $u_n > M$."
8. Yes, vacuously: no real x has $x^2 = -1$, so there is no counterexample.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.