

TMUA

Paper 2

Reasoning with numbers and counting

Free lesson notes

Covers	2.5 parity, divisibility, primes, remainders, pigeonhole, counting, invariants, dat
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

This is the widest lesson in Paper 2, because the TMUA hides logic inside whatever it likes: a parity argument, a remainder, a pigeonhole count, a path on a grid, a process that repeats. What unites these questions is that a calculation by brute force is hopeless or very slow, and a one-line structural idea makes the answer almost immediate. Spot the structure and you gain minutes; miss it and a question that should take ninety seconds eats five.

We build a small toolkit: arithmetic with remainders; the prime structure “ $6k \pm 1$ ”; the pairing method for pigeonhole questions that ask for the smallest number that *forces* something; the three counting moves that cover nearly every grid or code question; *invariants* (a quantity a process can never change) with the chessboard-colouring trick; cycles in repeated rules; and a gentle section on bounds and on means and medians. Thirteen worked examples, each answer verified by brute force.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **Parity, remainders and primes:** “which of these equations has no integer solutions” (squares modulo 4), “which expression is always divisible by 12 for a prime $p > 3$ ”, remainders of large powers.
- **Pigeonhole:** the *smallest* n that *forces* two chosen terms to sum to a target, or two numbers to differ by a multiple of m .
- **Counting:** routes on a grid (sometimes avoiding a junction), arrangements and codes, and “mountain profile” sequences of up- and down-steps that never go below the ground.
- **Iterated rules:** a function with different rules for odd and even inputs applied over and over; find the cycle, then jump to the 1000th term.
- **Spirals and patterns:** the coordinates of the n th number in a square spiral, using the perfect squares on the diagonal.
- **Invariants and colouring:** an operation that preserves parity; dominoes on a chessboard with squares removed.
- **Standard form, means and medians:** inequalities from given ranges, and the least possible range of a list with a given mean and median.

Parity, remainders and primes

Work with remainders: to compute something modulo m , replace every number by its remainder first, then add, subtract and multiply, reducing as you go. Parity is the case $m = 2$. Three facts are worth memorising:

- A square is 0 or 1 modulo 4, and 0 or 1 modulo 3. (So $x^2 + y^2$ is never 3 modulo 4, and $x^2 - y^2$ is never 2 modulo 4.)
- Every prime $p > 3$ has the form $6k \pm 1$ (it is odd and not a multiple of 3). Then $p^2 - 1 = (6k \pm 1)^2 - 1 = 12k(3k \pm 1)$, and $k(3k \pm 1)$ is always even, so $24 \mid p^2 - 1$.
- Powers of a number modulo m repeat in a cycle, so reduce the exponent by the cycle length.

Strategy: reduce, then decide

Before you try any large computation, ask *which remainder do I actually need?* Replace each number by its remainder, tabulate the possibilities (there are only 2, 3 or 4), and see which combinations are allowed. A target that is not on the list is unreachable, with no search needed.

Worked example: equations with no integer solutions

Question. Which of the following equations, in integers x and y , has/have **no** solution? I: $x^2 + y^2 = 1003$. II: $x^2 + y^2 = 1010$. III: $x^2 - y^2 = 1010$.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. I: squares are 0 or 1 modulo 4, so $x^2 + y^2$ is 0, 1 or 2 modulo 4. But $1003 = 4 \times 250 + 3$ leaves 3: no solution. II: $1010 \equiv 2 \pmod{4}$ is possible, and indeed $31^2 + 7^2 = 961 + 49 = 1010$: it has solutions. III: $x^2 - y^2$ is 0, 1 or 3 modulo 4 (squares 0 or 1: $0 - 0, 0 - 1, 1 - 0, 1 - 1$ give 0, 3, 1, 0), never 2; and $1010 \equiv 2$: no solution. So I and III have no solution: F.

Examiner's eye. The question asks for the equations with *no* solution; read the polarity. Option II has solutions even though 1010 is also not a sum of two *obvious* squares. Modular arithmetic only rules things out; a solution has to be exhibited.

Worked example: primes greater than 3

Question. For every prime $p > 3$, which of the following is/are divisible by 12? I: $p^2 - 1$. II: $p^2 + 11$. III: $p^2 + 3$.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. Write $p = 6k \pm 1$. Then $p^2 - 1 = 12k(3k \pm 1)$, a multiple of 12 (indeed of 24). I: yes. II: $p^2 + 11 = (p^2 - 1) + 12$, a multiple of 12: yes. III: $p^2 + 3 = (p^2 - 1) + 4$, which leaves remainder 4 when divided by 12: no (try $p = 5$: 28). So I and II only: E.

Examiner's eye. One test value (here $p = 5$: 24, 36, 28) already kills option III and supports I and II. Use a number to test, then the $6k \pm 1$ form to prove.

Sums of consecutive integers

k consecutive integers starting at a add up to $a + (a + 1) + \dots + (a + k - 1) = \frac{k(2a + k - 1)}{2}$. The triangular numbers are $T_n = 1 + 2 + \dots + n = \frac{n(n + 1)}{2}$, with the useful identities $T_n + T_{n+1} = (n + 1)^2$ and $8T_n + 1 = (2n + 1)^2$. A sum of k consecutive integers has the two factors k and $2a + k - 1$ in the product $k(2a + k - 1) = 2 \times \text{sum}$; these factors always have *opposite parity*, because they differ by $2a - 1$, which is odd. That one fact decides which numbers are possible.

Worked example: which numbers are sums?

Question. Which of the following can be written as the sum of two or more consecutive positive integers? I: 15. II: 16. III: 20.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. I: $15 = 7 + 8 = 4 + 5 + 6 = 1 + 2 + 3 + 4 + 5$: yes. III: $20 = 2 + 3 + 4 + 5 + 6$: yes. II: suppose 16 is a sum of $k \geq 2$ consecutive integers from $a \geq 1$. Then $k(2a + k - 1) = 32$. The two factors have opposite parity, so one of them is odd; but the only odd divisor of 32 is 1, forcing $k = 1$ (not allowed) or $2a + k - 1 = 1$ (impossible since $2a + k - 1 \geq 3$). So 16 is not such a sum (no power of 2 is). So I and III: F.

Examiner's eye. The opposite-parity fact is the whole idea: a sum of consecutive integers has an odd factor greater than 1 hidden in it, so a power of 2 (all of whose factors are even) cannot be one. Triangular identities, such as $8T_n + 1 = (2n + 1)^2$, come from the same product.

Pigeonhole and the pairing method

Pigeonhole principle: if more than m objects are placed in m boxes, some box has at least two. The TMUA version asks for the *smallest number of objects that forces* a pair with some property. The answer is always “the largest set that avoids the property”, plus 1.

Strategy: pair, count the worst case, add one

(1) Split the objects into **pairs** (or boxes) so that any two objects in the same pair have the property you want to force. (2) Leftovers that belong to no pair are “free”: they can be chosen without ever creating the property. (3) The largest *bad* selection takes one object from each pair plus every free object. (4) The smallest number that forces the property is that count plus 1. Check the pairing: each object must appear in exactly one pair or be free.

Worked example: forcing a sum

Question. The set $\{4, 7, 10, \dots, 91\}$ consists of the numbers $3k + 1$ for $k = 1, 2, \dots, 30$. What is the smallest n such that any n distinct numbers chosen from this set must include two that sum to 74?

Solution. Two terms $3j + 1$ and $3k + 1$ sum to 74 when $3(j + k) = 72$, that is $j + k = 24$. With $j \neq k$ the pairs are $(1, 23), (2, 22), \dots, (11, 13)$: 11 **pairs**. Not paired: $k = 12$ (it would need to pair with itself: $37 + 37$ uses the same term twice) and $k = 24, \dots, 30$ (partner $24 - k \leq 0$): that is $1 + 7 = 8$ free numbers. The largest bad selection takes one from each pair and all the free numbers: $11 + 8 = 19$. So any 20 numbers must contain a pair summing to 74: $n = 20$.

Examiner's eye. Two traps: the term 37 is its own partner, so it is free, and the large terms have no partner, so they are free too. Counting the pairs alone, $11 + 1 = 12$, would be wrong.

Worked example: sum or difference

Question. What is the smallest n such that among any n integers there must be two whose sum or whose difference is divisible by 10?

Solution. Sort integers by their remainder on division by 10. Two integers with the *same* remainder have a difference divisible by 10; two with remainders adding to 10 (like 3 and 7) have a sum divisible by 10. So the boxes are $\{0\}, \{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}, \{5\}$: **six boxes**, and two integers in one box have the property. Hence among 7 integers two share a box. But 6 integers are not enough: 0, 1, 2, 3, 4, 5 lie in six different boxes and no two have sum or difference divisible by 10. So $n = 7$.

Examiner's eye. The sharpness check, a set of six that fails, is what shows 7 is the *smallest*. Always exhibit the bad selection of one fewer.

Counting: grids, codes and mountain profiles

Three moves cover almost everything. **Multiply** independent choices. **Subtract** (count everything, then remove the unwanted). **Choose positions:** a route of r steps right and u steps up is a string of $r + u$ moves, and it is fixed by choosing which u of them go up: $\binom{r+u}{u}$ routes. To avoid a junction, count all routes and subtract those through it (routes to the junction times routes from it).

Worked example: avoiding a junction

Question. A walker goes from the corner $(0, 0)$ to $(5, 4)$ of a rectangular grid, moving one unit right or one unit up each step. How many routes avoid the junction $(2, 2)$?

Solution. All routes: $\binom{9}{4} = 126$. Routes through $(2, 2)$: $\binom{4}{2} = 6$ to get there, then 3 right and 2 up to finish: $\binom{5}{2} = 10$, so $6 \times 10 = 60$. Routes avoiding it: $126 - 60 = 66$.

Examiner's eye. The subtraction is safe because a route passes through $(2, 2)$ at most once. If you must avoid several points, subtract with care for routes that pass through two of them.

Worked example: mountain profiles

Question. A mountain profile is a sequence of steps, each $+1$ (up) or -1 (down), starting at height 0 and finishing at height 0, never going below 0. How many profiles have 8 steps? How many of these touch height 0 only at the start and the end?

Solution. With 4 ups and 4 downs there are $\binom{8}{4} = 70$ sequences in total. Remove the ones that dip below 0. A bad sequence first reaches height -1 at some step; reflect everything *up to that step*. The result is a sequence with 3 ups and 5 downs, and every such sequence arises exactly once: $\binom{8}{3} = 56$ bad ones. So there are $70 - 56 = 14$ profiles. For those that stay strictly above 0 in between: the profile must start up, end down, and the 6 steps between form a profile of height ≥ 0 for 6 steps, i.e. a 6-step profile shifted up one: that count is $\binom{6}{3} - \binom{6}{2} = 5$. So 5 touch the ground only at the ends.

Examiner's eye. For small step counts, listing is safe and fast: 2 steps give 1 profile, 4 steps give 2, 6 steps give 5, 8 steps give 14. The reflection trick avoids listing when the numbers grow.

Iterated rules, invariants and patterns

Iterated rules. A rule applied again and again on integers often falls into a cycle. Compute terms until one repeats, note the **cycle length** c and where it starts, then use the remainder of the position on division by c to jump to a far-away term.

Invariants. If every allowed move leaves some quantity unchanged (the parity of a total, a remainder, the number of squares of each colour), that quantity decides what is possible without exploring the moves. The check is to confirm that the quantity really is unchanged by *every* move, then to compare it with the target.

Colouring. Colour a board like a chessboard; a 1×2 domino always covers one square of each colour.

Worked example: find the cycle

Question. A sequence has $a_1 = 5$ and, for $k \geq 1$, $a_{k+1} = \frac{a_k}{2}$ if a_k is even, and $a_{k+1} = a_k + 7$ if a_k is odd. Find a_{1000} .

Solution. Compute: $5 \rightarrow 12 \rightarrow 6 \rightarrow 3 \rightarrow 10 \rightarrow 5$. So $a_1, \dots, a_5 = 5, 12, 6, 3, 10$ and then $a_6 = 5 = a_1$: the sequence repeats with cycle length 5 from the very start. Hence $a_k = a_{k \bmod 5}$ with a_0 taken as a_5 . Since 1000 is a multiple of 5, $a_{1000} = a_5 = 10$.

Examiner's eye. Make the position bookkeeping explicit: write the cycle with its positions 1 to 5, and note that a multiple of 5 lands on the *last* entry, not the first.

Worked example: an invariant

Question. The numbers $1, 2, 3, \dots, 9$ are written on a board. A move erases two numbers a, b and writes $|a - b|$ in their place. After eight moves one number remains. Which of the following could it be? I: 9. II: 6. III: 10.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

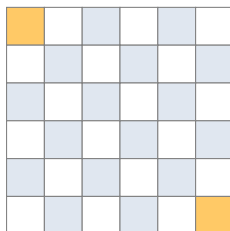
Solution. Look for what a move preserves. Replacing a, b by $|a - b|$ changes the total by $(a + b) - |a - b| = 2 \min(a, b)$, an even number. So the *parity* of the total never changes. The initial total is $1 + 2 + \dots + 9 = 45$, odd, so the final number is odd: II (6) is impossible. Also $|a - b| \leq \max(a, b)$ so no number ever exceeds 9: III (10) is impossible. I: 9 can be reached. Leave the 9 alone and shrink $1, \dots, 8$ to 0: $(1, 2) \rightarrow 1$, $(3, 4) \rightarrow 1$, $(5, 6) \rightarrow 1$, $(7, 8) \rightarrow 1$; then $(1, 1) \rightarrow 0$ twice; then $(0, 0) \rightarrow 0$; and finally $(9, 0) \rightarrow 9$. That is $4 + 2 + 1 + 1 = 8$ moves. So I only: B.

Examiner's eye. Parity says 9 *might* be possible; it does not prove it. For a "could it be" question you need both an invariant to rule values out and an explicit sequence to rule one in.

Worked example: dominoes on a board

Question. A 6×6 chessboard has two squares removed. Can the remaining 34 squares be covered exactly by 1×2 dominoes (no overlaps, none sticking out)? Consider I: the two opposite corners (top left and bottom right). II: the two ends of the top row. III: the squares in the first row second column, and in the second row first column. Which can be covered? (The picture shows the board for option I, with the removed squares shaded.)

- A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

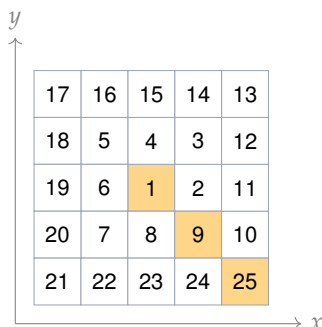


Solution. Each domino covers one dark and one light square. The full board has 18 of each. If the two removed squares have the *same* colour, the remainder has 16 of one colour and 18 of the other, so no tiling exists. I: opposite corners (the picture) have the same colour: impossible. III: the squares in row 1, column 2 and row 2, column 1 both have row + column = 3, odd: same colour, impossible. II: the ends of the top row, columns 1 and 6, have row + column equal to 2 and 7: different colours, and a tiling exists: two horizontal dominoes cover columns 2 to 5 of the top row, and each of the other five rows holds three horizontal dominoes. So II only: C.

Examiner's eye. The colouring argument can only *rule out*. When colours are balanced it does not by itself prove a tiling exists: you must exhibit one (or know the theorem that two squares of opposite colours can always be removed from a rectangular board).

Worked example: a spiral

Question. The positive integers are written in a square spiral: 1 at (0,0), then 2 at (1,0), 3 at (1,1), 4 at (0,1), and so on anticlockwise, as shown. Which point has the number 2000 written on it?



Solution. Look at the shaded numbers 1, 9, 25: odd squares on the diagonal, at (0,0), (1, -1), (2, -2). In general $(2m + 1)^2$ sits at $(m, -m)$. The nearest odd square above 2000 is $45^2 = 2025$ ($m = 22$), at (22, -22). The numbers just before 2025 run *leftwards* along the bottom row $y = -22$: $2025 - j$ is at $(22 - j, -22)$ for $j = 0, 1, \dots, 44$. Now $2000 = 2025 - 25$, so it lies at $(22 - 25, -22) = (-3, -22)$.

Examiner's eye. Anchor on the diagonal squares, then count back along one side. The side has length 44 here, so $j = 25$ is safely within it; always check that your step count does not exceed the side.

Standard form, bounds, and a little data

Standard form $a \times 10^k$ with $1 \leq a < 10$: to compare, compare the powers of ten first, then the numbers a . For products and quotients, multiply the a 's and add the powers (or divide and subtract). For bounds, a quantity between two values is largest or smallest at a *corner*: for pq and p/q with positive p, q , use p and q at their extreme values (largest p with smallest q for the biggest quotient).

Mean, median and range. Sort the list. The median is the middle value, the mean is the total over the count, and the range is the largest minus the smallest. To minimise the range, move the values as close together as the constraints allow.

Worked example: bounds in standard form

Question. The positive numbers p and q satisfy $2 \times 10^5 \leq p \leq 5 \times 10^5$ and $4 \times 10^{-2} \leq q \leq 8 \times 10^{-2}$. Which of the following must be true? I: $pq \geq 8 \times 10^3$. II: $\frac{p}{q} \leq 1.25 \times 10^7$. III: $p - q > 2 \times 10^5$.

A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. I: the smallest product uses both smallest values: $2 \times 10^5 \times 4 \times 10^{-2} = 8 \times 10^3$, so $pq \geq 8 \times 10^3$: true. II: the largest quotient has the largest p and smallest q : $\frac{5 \times 10^5}{4 \times 10^{-2}} = 1.25 \times 10^7$, so true. III: $p - q$ is smallest when $p = 2 \times 10^5$ and q is largest, 0.08, giving $199\,999.92 < 2 \times 10^5$: not forced. So I and II only: E.

Examiner's eye. Option III is the strictness trap: even the "obvious" $p - q > 2 \times 10^5$ fails because q can be positive when p is at its lower end.

Worked example: least possible range

Question. A list of five positive integers has median 6 and mean 8. What is the least possible range?

Solution. Write the list as $a \leq b \leq 6 \leq d \leq e$ with total 40, so $a + b + d + e = 34$. To make $e - a$ small we want a large and e small. The ceiling is $a \leq b \leq 6$ so $a + b \leq 12$, which forces $d + e \geq 22$. The best case is $a = b = 6$ and $d = e = 11$, which gives total $6 + 6 + 6 + 11 + 11 = 40$ and range $11 - 6 = 5$. (Lowering a to 5 with $b = 6$ needs $d + e = 23$, so $e \geq 12$ and the range is at least 7.)

Examiner's eye. The structure is: median fixes the middle, the mean fixes the total, and the range is controlled by the extremes. Push each pair of unknowns as close to the median as the total allows.

Habits that win marks

- Look for the structure before you calculate: parity, a remainder, a pairing, a symmetry, an invariant.
- For modular questions, reduce first and tabulate the possible remainders; a target not on the list is impossible.
- Every prime greater than 3 is $6k \pm 1$; among three consecutive integers one is a multiple of 3; among two, one is even.
- "Smallest n that forces": pair up, count the worst bad selection, add 1, and exhibit the bad selection for $n - 1$. Do not forget unpaired free elements and self-paired elements.
- For routes, count all and subtract the unwanted; for codes, build the sequence step by step and look for a reflection or a pattern in small cases.
- For iterated rules, write the first dozen terms, spot the cycle and its start, then use remainders on the position.
- An invariant rules a value out; to rule one in you must construct it. Check that the quantity really is preserved by every move.
- With ranges and means, consider the corners, and always check the strict and non-strict end points.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. What is the remainder when $7^{2025} + 3^{2025}$ is divided by 10?
2. How many four-digit numbers have every digit odd?
3. True or false: the sum of the squares of two odd integers is divisible by 4.
4. Six integers are chosen. Must two of them differ by a multiple of 5?
5. Show that $T_n + T_{n+1} = (n+1)^2$, where $T_n = \frac{n(n+1)}{2}$.
6. A route from one corner of a 3×3 grid of squares to the opposite corner uses only right and up moves. How many routes are there?
7. Numbers $1, 2, \dots, 10$: what is the smallest number you must choose to guarantee that two of them sum to 11?
8. A 6×6 board has two squares of the same colour removed. Why can the rest not be tiled by dominoes?

Answers to the self-check

1. Powers of 7 end 7, 9, 3, 1 and powers of 3 end 3, 9, 7, 1, both with cycle 4. $2025 = 4 \times 506 + 1$, so the last digits are 7 and 3, summing to 10: remainder 0.
2. $5 \times 5 \times 5 \times 5 = 625$.
3. False: $1^2 + 3^2 = 10$. Each odd square is 1 modulo 4, so the sum is 2 modulo 4.
4. Yes: only 5 possible remainders on division by 5, so two of the six share one (pigeonhole).
5. $\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{(n+1)(2n+2)}{2} = (n+1)^2$.
6. Three rights and three ups: $\binom{6}{3} = 20$.
7. Pairs (1, 10), (2, 9), (3, 8), (4, 7), (5, 6): five pairs, no free numbers. The worst bad selection has 5, so 6 numbers force a pair.
8. Each domino covers one square of each colour, so a tiling needs equally many of each. Removing two of one colour leaves 16 of that colour and 18 of the other.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.