

TMUA

Paper 2

Proof and counterexamples

Free lesson notes

Covers	2.3 direct proof, contradiction, exhaustion, counterexamples
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

A proof shows a statement is true in *every* case. A counterexample shows it is false with a *single* case. That asymmetry is the engine of Paper 2: no number of examples proves a “for all” claim, yet one example destroys it. The TMUA asks you to use this in three ways: pick which of three given examples is a counterexample, judge which proof method fits a claim, and spot a statement that sounds proved but is only checked.

This lesson teaches the four methods (counterexample, direct proof, cases, contradiction) with the actual patterns that recur: expressions that look prime-generating, claims about derivatives, parity and remainder arguments, and irrationality. It gives a systematic way to *hunt* for a counterexample, and a short list of proof skeletons you can adapt. There are eleven worked examples, all verified by computer.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **Which of I, II, III is a counterexample?** A universal claim (“for all positive integers n , $n^2 + n + 11$ is prime”) and three candidate values; you must test each carefully, including whether it satisfies the hypothesis.
- **Prime-producing expressions:** $n^2 + n + c$, $2p + 1$, $10p^2 \pm 1$ and similar, which work for many small values and then fail. Also “every integer greater than... is a sum of two primes / non-primes”.
- **Calculus claims:** “if $f'(x) > 0$ for all x then $f(x) > 0$ ” (false), and similar mismatches between a derivative or an integral and the sign of the function.
- **Averages and speeds:** “the average speed over two equal-distance halves is the mean of the two speeds” (false).
- **Parity and divisibility proofs:** consecutive integers, squares modulo 3 or 4, “ $n^5 - n$ is divisible by...”, and complete-the-proof questions.
- **Contradiction:** irrationality, “no integer solutions”, infinitely many primes; often the question asks which assumption to make or which statement ends the proof.

Counterexamples: how to disprove a “for all”

A statement “for every x in some set, $P(x) \Rightarrow Q(x)$ ” is disproved by one x where $P(x)$ is true and $Q(x)$ is false. Two conditions, not one: a candidate that fails the *hypothesis* is irrelevant, however badly it behaves. A candidate that satisfies both hypothesis and conclusion proves nothing.

Strategy: hunt systematically

When you must *find* a counterexample, do not wander. Try, in order: (1) the smallest values (0, 1, 2, 3); (2) negatives, and zero; (3) fractions between 0 and 1 (squares and square roots behave differently there); (4) equal values ($a = b$) and boundary values; (5) the first value where a “pattern” could break (for $n^2 + n + c$ that is often $n = c - 1$ or $n = c$, where a factor c appears). Pick the example that makes the conclusion fail, not just any example.

Worked example: a prime-producing expression

Question. Claim: “for every positive integer n , $n^2 + n + 11$ is prime.” Which of the following is/are a counterexample? I: $n = 5$. II: $n = 7$. III: $n = 10$.

- A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. Compute each. $n = 5$: $25 + 5 + 11 = 41$, prime. $n = 7$: $49 + 7 + 11 = 67$, prime. $n = 10$: $100 + 10 + 11 = 121 = 11^2$, not prime. Only III is a counterexample: **D**.

Examiner’s eye. The famous $n^2 + n + 41$ works for $n = 0, \dots, 39$ and fails at $n = 40$ ($40^2 + 40 + 41 = 41^2$). Thirty-nine successes prove nothing. The failure comes where the added constant becomes a factor: $n^2 + n + c$ is divisible by c when $n = c - 1$ or $n = c$.

Worked example: sums of two primes

Question. Claim: “every integer greater than 1 is either prime or the sum of two primes.” Which of the following is/are a counterexample? I: 21. II: 27. III: 35.

- A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. A counterexample is a number that is *not* prime and is *not* a sum of two primes. For an odd number n , a sum of two primes must be $2 + (n - 2)$ (one prime must be even, so equal to 2), so n works only if $n - 2$ is prime. I: $21 = 2 + 19$, fine. II: 27 is not prime and 25 is not prime, so 27 is not a sum of two primes: counterexample. III: 35 is not prime and 33 is not prime: counterexample. So II and III: **G**.

Examiner’s eye. The parity observation (an odd sum of two primes must contain the prime 2) turned a search over many pairs into a single primality test of $n - 2$.

Worked example: a calculus claim

Question. Claim: “if f is a function with $f'(x) > 0$ for all real x , then $f(x) > 0$ for all real x .” Which of the following is/are a counterexample? I: $f(x) = x - 1$. II: $f(x) = x^3 + 5$. III: $f(x) = e^x - 3$.

- A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. A counterexample needs $f' > 0$ *everywhere* and $f < 0$ *somewhere*. I: $f' = 1 > 0$, and $f(0) = -1 < 0$. Counterexample. II: $f(-2) = -3 < 0$, but $f'(x) = 3x^2$ is 0 at $x = 0$, so the hypothesis fails. Not a counterexample. III: $f'(x) = e^x > 0$ everywhere and $f(0) = -2 < 0$. Counterexample. So I and III: **F**.

Examiner’s eye. Option II is the trap: it is negative somewhere, but it has a stationary point, so it breaks the hypothesis. “Increasing” and “positive” are unrelated: $f' > 0$ only says the graph climbs, not where it starts.

Worked example: average speed

Question. Claim: “a journey is made in two halves of equal distance, at speeds a and b (km/h) respectively. The average speed for the whole journey is $\frac{1}{2}(a + b)$.” Which of the following is/are a counterexample? I: $a = b = 30$. II: $a = 40$, $b = 60$. III: $a = 20$, $b = 60$.

- A none of them B I only C II only D III only
E I and II only F I and III only G II and III only H I, II and III

Solution. Average speed is total distance over total time. For distance d each way the time is $\frac{d}{a} + \frac{d}{b}$, so the average is $\frac{2d}{d/a + d/b} = \frac{2ab}{a + b}$. I: $\frac{2 \cdot 900}{60} = 30 = \frac{1}{2}(30 + 30)$: no. II: $\frac{2 \cdot 2400}{100} = 48 \neq 50$: counterexample. III: $\frac{2 \cdot 1200}{80} = 30 \neq 40$: counterexample. So II and III: **G**.

Examiner’s eye. Equal *distances* means more time is spent at the slower speed, so the true average is pulled below the arithmetic mean. The arithmetic mean is right only if the *times* are equal, or if $a = b$ (option I).

Common mistakes

- Citing an example that satisfies the conclusion as “evidence” for a universal claim, or a non-example as a counterexample because it “looks bad”. Check the hypothesis too.
- Believing that many successes make a claim likely to be true. $n^2 + n + 41$ succeeds 40 times in a row.
- Giving a counterexample for the *converse* when the question is about the original statement.

Direct proof and algebraic proof

In a **direct proof** you start from the hypothesis and reason, step by step, to the conclusion, using a general object (not a particular number). For integers the standard general objects are: an even number $2k$, an odd number $2k + 1$, a multiple of 3 as $3k$, and consecutive integers $n, n + 1, n + 2, \dots$. Algebra then does the work. Three tricks recur: *factorise* to expose a multiple, *pair* factors symmetrically (outer with outer, inner with inner), and use the fact that **among two consecutive integers one is even** and **among three consecutive integers one is a multiple of 3**.

Worked example: four consecutive integers

Question. Prove that the product of four consecutive integers, plus 1, is a perfect square.

Solution. Let the integers be $n, n + 1, n + 2, n + 3$. Pair the outer two and the inner two: $n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$. Write $m = n^2 + 3n$. Then

$$n(n + 1)(n + 2)(n + 3) + 1 = m(m + 2) + 1 = m^2 + 2m + 1 = (m + 1)^2 = (n^2 + 3n + 1)^2.$$

This is the square of an integer. (Check: $n = 2$: $2 \cdot 3 \cdot 4 \cdot 5 + 1 = 121 = 11^2$ and $4 + 6 + 1 = 11$.)

Examiner's eye. The pairing made the two factors differ by exactly 2, which is the shape $m(m + 2) = (m + 1)^2 - 1$. When four or more brackets appear, look for a pairing with a common part.

Worked example: odd squares

Question. Prove that the square of any odd integer leaves remainder 1 on division by 8.

Solution. An odd integer is $2k + 1$. Then

$$(2k + 1)^2 = 4k^2 + 4k + 1 = 4k(k + 1) + 1.$$

Now $k(k + 1)$ is a product of two consecutive integers, so it is even: $k(k + 1) = 2j$. Hence $(2k + 1)^2 = 8j + 1$, remainder 1 on division by 8.

Examiner's eye. A useful consequence: the difference of two odd squares is a multiple of 8, and an odd square can never be 3 more than a multiple of 4.

Proof by cases (exhaustion)

If a statement is about all integers, split the integers into a small number of classes that together cover everything, and prove the statement in each class. The classes are usually the remainders modulo a small number: even/odd, or 0, 1, 2 modulo 3, or 0, 1, 2, 3 modulo 4. The proof is complete only if the cases **cover every integer** and **each case is fully proved**. A table of squares of the remainders is a powerful tool:

$n \bmod 4$	0	1	2	3
$n^2 \bmod 4$	0	1	0	1

$n \bmod 3$	0	1	2
$n^2 \bmod 3$	0	1	1

So a square is 0 or 1 modulo 4, and 0 or 1 modulo 3; and a square is 0 (mod 3) *only* when the number itself is a multiple of 3.

Worked example: a sum of two squares

Question. Prove that if a and b are integers and $a^2 + b^2$ is divisible by 3, then both a and b are divisible by 3.

Solution. Squares are 0 or 1 modulo 3, and a square is 0 exactly when the number is a multiple of 3. Consider the possible remainders of a^2 and b^2 : $0 + 0 = 0, 0 + 1 = 1, 1 + 0 = 1, 1 + 1 = 2$. Only $0 + 0$ gives remainder 0, so $a^2 \equiv b^2 \equiv 0$, and hence $3 \mid a$ and $3 \mid b$.

Examiner's eye. The four remainder combinations are the whole proof: four cases, one table. It also shows that $3 \mid a^2 + b^2$ forces $9 \mid a^2 + b^2$ (write $a = 3s, b = 3t$).

Worked example: $n^5 - n$

Question. Prove that $n^5 - n$ is divisible by 30 for every integer n .

Solution. Factorise: $n^5 - n = n(n^4 - 1) = (n - 1)n(n + 1)(n^2 + 1)$. Since $30 = 2 \times 3 \times 5$ and these are pairwise coprime, prove divisibility by each.

By 2 and 3: $(n - 1)n(n + 1)$ is three consecutive integers, so it contains a multiple of 2 and a multiple of 3.

By 5 (cases on $n \bmod 5$): if $n \equiv 0, 1$ or 4 , then $n, n - 1$ or $n + 1$ is a multiple of 5. If $n \equiv 2$ or 3 , then $n^2 \equiv 4$ or $9 \equiv 4$, so $n^2 + 1 \equiv 5 \equiv 0$. These five cases cover every integer, so 5 always divides.

Hence $30 \mid n^5 - n$.

Examiner's eye. Factorising first turned a power into a product, and each prime factor of 30 then needed only a few cases. Always ask whether the divisor splits into coprime pieces.

Proof by contradiction

To prove a statement S by contradiction, **assume S is false** and show that this leads to something impossible (a statement that is false whatever happens). Then S cannot be false, so it is true. It is the natural method for negative statements: "irrational", "no integer solution", "there is no largest...", "cannot be written as...". The first step is therefore always to write the negation correctly, and the last is to state which fact has been contradicted.

Worked example: an irrationality proof

Question. Prove that $\sqrt{6}$ is irrational.

Solution. Suppose $\sqrt{6}$ is rational. Then $\sqrt{6} = \frac{p}{q}$ with positive integers p, q with no common factor. Squaring, $p^2 = 6q^2$, so p^2 is even, so p is even: $p = 2r$. Then $4r^2 = 6q^2$, i.e. $2r^2 = 3q^2$. Now $3q^2$ is even, so q^2 is even, so q is even. So p and q are both even, contradicting "no common factor". Hence $\sqrt{6}$ is irrational.

Examiner's eye. The "lowest terms" clause is the point: without it there would be no contradiction. The same skeleton proves $\sqrt{n}$ irrational whenever n is not a perfect square.

Worked example: no integer solutions

Question. Prove that there are no integers m, n with $m^2 - n^2 = 2$.

Solution. Suppose there are. Then $(m - n)(m + n) = 2$. The two factors differ by $2n$, which is even, so they have the same parity. If both are odd, their product is odd, not 2. If both are even, their product is a multiple of 4, not 2. Both cases are impossible, a contradiction. So no such integers exist. (The same argument shows that a difference of two squares is never 2 more than a multiple of 4.)

Examiner's eye. Parity of the two factors, not their exact values, was the key. For problems of the form "product of two integers equals a small number" check whether the factors can have the required parity.

Worked example: infinitely many primes

Question. Outline the proof that there are infinitely many primes, and explain why $p_1 p_2 \cdots p_k + 1$ need not itself be prime.

Solution. Suppose the primes are exactly $p_1, \dots, p_k$. Let $N = p_1 p_2 \cdots p_k + 1 > 1$. Then N has at least one prime factor q , and q must be one of $p_1, \dots, p_k$. But each p_i divides $N - 1$, so p_i cannot divide N (it would divide $N - (N - 1) = 1$). Contradiction: q is a prime not in the list.

N need not be prime. For the first six primes, $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 + 1 = 30031 = 59 \times 509$. It is composite, but its prime factors 59 and 509 are new primes.

Examiner's eye. The contradiction claim is "a prime outside the list", not " N is a new prime". Questions often test exactly this distinction.

Choosing the method

Read the *shape* of the claim before you decide how to attack it:

Pick a method from the shape

Claim shape	First move
"for all $n \dots$ " (looks true)	Direct proof with a general object ($2k, n, n+1, \dots$), or cases on remainders.
"for all $n \dots$ " (looks suspicious)	Test 0, 1, 2, negatives, fractions, boundary values. One failure is a counterexample.
"there exists $\dots$ "	Produce one example and check it.
"there is no $\dots$ ", "irrational", "cannot"	Contradiction: assume it <i>can</i> , and chase the parity/factors.
"if P then Q " hard to start	Try the contrapositive: assume not Q , prove not P .
Complicated condition on integers	Reduce modulo a small number and tabulate residues.

Common mistakes

- Proving the *converse* by accident: showing " n even $\Rightarrow n^2$ even" when the claim is " n^2 even $\Rightarrow n$ even".
- Using a specific number ($n = 6$) as the "general" integer in a proof.
- In a contradiction proof, forgetting to name the contradiction, or assuming the wrong negation (the negation of "for all" is "there exists").

Habits that win marks

- One example can disprove a universal claim; no number of examples can prove it. Never let a "works for $n = 1, 2, 3$ " option persuade you.
- A counterexample must satisfy the *hypothesis* and break the *conclusion*. Check both for each candidate.
- Hunt in order: small values, zero, negatives, fractions, equal values, the first place a pattern could break.
- For parity and divisibility, set up the general object ($2k, 2k+1, n(n+1)$, remainders mod 3 or 4) before you compute anything.
- For "irrational", "no solutions", "cannot", use contradiction: write the negation first.
- If proving $P \Rightarrow Q$ is awkward, try the contrapositive. Do not accidentally prove the converse.
- In a proof by cases, say how the cases cover everything and check that every case is finished.

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. Disprove: "the sum of two irrational numbers is irrational".
2. Which method suits "prove that $\sqrt{3}$ is irrational"?
3. Prove that the product of an even integer and any integer is even.
4. Give a counterexample to " $n^2 > n$ for every integer n ".
5. Give a counterexample to "if $a > b$ then $a^2 > b^2$ ".
6. Give a counterexample to " $2^n + 1$ is prime for every positive integer n ".
7. Prove that the sum of any three consecutive integers is divisible by 3.
8. Prove, using the contrapositive, that if n^2 is odd then n is odd.

Answers to the self-check

1. $\sqrt{2}$ and $-\sqrt{2}$ are irrational and their sum is 0, which is rational.
2. Contradiction: assume $\sqrt{3} = p/q$ in lowest terms; then $p^2 = 3q^2$ forces $3 \mid p$, then $3 \mid q$: a common factor.
3. The even integer is $2k$ and the other is m . Then $2k \cdot m = 2(km)$, which is even.
4. $n = 1$ gives $1 > 1$, false. ($n = 0$ also fails.)
5. $a = 1, b = -2$: $a > b$ but $a^2 = 1 < 4 = b^2$.
6. $n = 3$: $2^3 + 1 = 9 = 3 \times 3$. (It works for $n = 1, 2, 3$ and 5 ; and $n = 4$: 17.)
7. $(n - 1) + n + (n + 1) = 3n$.
8. Contrapositive: if n is even then n^2 is even. Indeed $n = 2k$ gives $n^2 = 4k^2 = 2(2k^2)$.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.