

TMUA

Paper 2

Spotting errors in arguments

Free lesson notes

Covers	2.4 finding the first error in a mathematical argument
Format	Lesson, worked examples, self-check
Price	Free

About this lesson

Every Paper 2 includes at least one question that shows a short solution or proof, six lines long, and asks where the *first* error occurs (or whether the argument is correct). Students find these slow and confusing, because the arguments are written to be convincing: the later lines are often *true*, the arithmetic is correct, and the mistake is a single illegal move buried in the middle.

The cure is a method. For every line you ask one question: *does this follow from the line above, and is the move legal for every value of the variable?* This lesson turns that into a routine, lists the dozen illegal moves that the TMUA keeps using (squaring, dividing by something that can be zero, losing a negative root, multiplying an inequality by a negative, a missing chain-rule factor, a non sequitur, a converse), and shows how to test a suspect line with one well-chosen number. There are eleven worked examples in the exact option format (A to F for the lines, G for “no error”), each one checked by a short program that referees the argument.

How to use this lesson. Read a section, then cover each worked example and try it before reading the solution. The TMUA gives you about 90 seconds a question on average, so after every idea ask two things: what is the quickest sound method, and what would a quick sketch or a test value tell me?

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The patterns to expect

What the TMUA keeps asking

- **Solve an equation or inequality, find the first error:** six lines, options “line (I) ... (VI)” or “the argument is correct”. The usual culprits: squaring, dividing by x (or by an expression that can be 0), taking a square root and keeping only the $+$ root, multiplying an inequality by something negative.
- **Proof with a hidden gap:** a proof by cases that misses a case, an “it follows that” that needs more, or a proof of the converse instead of the statement.
- **Calculus slips:** differentiating an integral with a variable limit and forgetting the chain-rule factor; using $f' > 0$ to claim $f > 0$.
- **Non sequitur:** “no real linear factors, so it is prime”, “the formula works for $n = 1, 2, 3$, so it always works”.
- **Correct arguments:** roughly one in six is flawless (usually containing a squaring or a cancellation that *is* legal). “No error” is a real answer.
- **State the error:** given a solution that ends with a wrong answer, choose the sentence that correctly describes what went wrong.

A method: check each step in the right direction

A solution to an equation is a chain of lines. Each step is one of three kinds:

Three kinds of step

Step	What it means
Equivalence ($\Leftrightarrow$)	Every solution of the line above is a solution of this line and vice versa. Adding the same thing to both sides, expanding, factorising, multiplying or dividing by a <i>non-zero constant</i> .
Implication only ($\Rightarrow$)	Everything that solves the line above still solves this one, but <i>extra</i> solutions may appear. Squaring both sides, multiplying by an expression that can be zero. Legal, but you must check candidates in the original at the end.
Invalid	Real solutions are <i>lost</i> , or a false inference is made. Dividing by something that can be 0; taking $\sqrt{a^2} = a$; multiplying an inequality by a quantity of unknown sign; " $ab = 6 \Rightarrow a = 6$ or $b = 6$ ".

Two consequences. (1) If a solution loses roots (an invalid step), the error is on *that* line, even if every later line looks perfect. (2) If a solution only used $\Rightarrow$ steps, the candidates it produces are *possible* solutions; a line that says "so both are solutions" without checking is the error, even though the algebra before it was legal.

Strategy: test the suspect line with a number

Do not trust a feeling that a line is wrong. Pick a value that makes the line above *true*, and see whether the line below is also true. For a division step, look for the value that makes the divisor 0. For an inequality step, try a negative value. A single value that satisfies one line and fails the next proves the step invalid. And if you think a step is illegal but cannot find such a value, you may have found a perfectly good step, so consider the answer "the argument is correct".

The illegal moves, with a quick test for each

Illegal move	Quick test value
dividing by x (or by $x - a$)	the value that makes it 0; roots are lost
squaring an equation, then not checking $\sqrt{a^2} = a$ or "take roots"	substitute each candidate; $\sqrt{x+6} = x$ gives the false root -2
$a < b \Rightarrow a^2 < b^2$	$a = -3$: $\sqrt{9} = 3$, not -3 ; keep $\pm$
$\frac{1}{a} > \frac{1}{b} \Rightarrow a < b$	$a = -3$, $b = -1$ (both negative) or $a = -3$, $b = 2$
multiplying an inequality by an expression of unknown sign	$a = 1$, $b = -1$ (opposite signs)
$pq = c$ ($c \neq 0$) $\Rightarrow p = c$ or $q = c$	a value that makes that expression negative
$pq = c$ ($c \neq 0$) $\Rightarrow p = c$ or $q = c$	$(x-1)(x+4) > 6$: the bracket values need not be 6
differentiating $\int_1^{g(x)}$ without $g'(x)$	compare with integrating first
"no integer-coefficient factors, so prime"	test the first few values; one composite kills it
converse for the statement	try a case where the converse fails ($n = 9$ odd, not prime)

Solving equations: lost roots and false roots

The two most common errors are opposites. Dividing by x *loses* the root $x = 0$; squaring *creates* roots that need removing. In both cases the arithmetic can be flawless.

Worked example: dividing by x

Question. A student solves $x^3 = 4x$:

- (I) $x^3 = 4x$.
- (II) $x^3 - 4x = 0$.
- (III) $x^2 - 4 = 0$, dividing through by x .
- (IV) $(x-2)(x+2) = 0$.
- (V) So $x = 2$ or $x = -2$.
- (VI) Check: $2^3 = 8 = 4 \times 2$ and $(-2)^3 = -8 = 4 \times (-2)$, so the solutions are $x = 2$ and $x = -2$.

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Lines (I) and (II) are fine ($\Leftrightarrow$). Line (III) divides by x , which is 0 for $x = 0$, and $x = 0$ is a solution of (II). So (III) has lost a root: it is not equivalent to (II). Lines (IV) to (VI) are true statements in themselves, and the check in (VI) is arithmetically correct, but the first illegal step is (III): **C**. The correct route factorises $x(x^2 - 4) = x(x - 2)(x + 2) = 0$, giving $x = 0, \pm 2$.

Examiner's eye. The final claim is also false, but it is not the *first* error. The question is about the first illegal move, not the first false line. Here lines (III) to (V) are all *true* statements; the error is in the step into (III).

Worked example: squaring

Question. A student solves $\sqrt{x+6} = x$:

- (I) Square both sides: $x + 6 = x^2$.
- (II) Rearrange: $x^2 - x - 6 = 0$.
- (III) Factorise: $(x - 3)(x + 2) = 0$.
- (IV) So $x = 3$ or $x = -2$.
- (V) Both of these values satisfy the original equation.
- (VI) Hence the equation has two solutions.

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Squaring is a legal $\Rightarrow$ step, so lines (I) to (IV) are all correct (each follows from the one above, and they are equivalent to each other). Line (V) is a *claim* that both candidates work in the *original* equation. Test $x = -2$: $\sqrt{-2+6} = \sqrt{4} = 2 \neq -2$. So (V) is false: **E**. Only $x = 3$ is a solution.

Examiner's eye. The mistake is not squaring. It is stopping after squaring. Whenever a solution squares, the last job is to substitute the candidates back.

Worked example: taking square roots

Question. A student solves $(2x - 1)^2 = (x + 4)^2$:

- (I) $(2x - 1)^2 = (x + 4)^2$.
- (II) Take square roots of both sides: $2x - 1 = x + 4$.
- (III) Hence $x = 5$.
- (IV) Check: $(2 \times 5 - 1)^2 = 81$ and $(5 + 4)^2 = 81$.
- (V) So $x = 5$ is a solution.
- (VI) Therefore $x = 5$ is the only solution.

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Line (II) is invalid: from $A^2 = B^2$ you may conclude $A = B$ or $A = -B$, not just $A = B$. The second possibility, $2x - 1 = -(x + 4)$, gives $3x = -3$, $x = -1$, and indeed $(-3)^2 = 9 = (3)^2$. So (II) lost a root. Lines (III) to (V) are correct statements, and (VI) is false, but the first error is (II): **B**.

Examiner's eye. "Take square roots" is a polite phrase for $\pm$. A square-root step that keeps only one sign is the same error as $\sqrt{a^2} = a$.

Worked example: stating the error

Question. A student solves $\frac{1}{x-2} + \frac{1}{x+2} = \frac{4}{x^2-4}$ by multiplying through by $(x-2)(x+2)$ to get $(x+2) + (x-2) = 4$, so $2x = 4$ and $x = 2$. Which statement is correct?

- A. The solution $x = 2$ is correct.
- B. The student should also have found $x = -2$.
- C. The value $x = 2$ makes a denominator zero, so it is not a solution and the equation has no solution.
- D. The equation is true for all real x .

Solution. Multiplying through by $(x - 2)(x + 2)$ is only an implication: it can introduce a candidate that makes the original denominators zero. Here $x = 2$ gives $\frac{1}{0}$, which is undefined. In fact $\frac{2x}{x^2 - 4} = \frac{4}{x^2 - 4}$ forces $x = 2$, which is excluded, so the equation has **no** solution: **C**. (Also $x = -2$ is excluded, so B is wrong.)

Examiner's eye. Whenever the unknown is in a denominator, list the forbidden values *before* you start and strike any candidate that hits one.

Inequalities and hidden assumptions

Inequalities are less forgiving than equations: multiplying or dividing by a negative reverses them, squaring is allowed only when both sides are non-negative, and a product being large does not make a factor large. The signs involved depend on the value of x , so a step that is right for $x > 0$ may be wrong for $x < 0$.

Worked example: a product and a constant

Question. A student solves $(x - 1)(x + 4) > 6$ for real x :

- (I) Expand: $x^2 + 3x - 4 > 6$.
- (II) Rearrange: $x^2 + 3x > 10$.
- (III) Factorise: $x(x + 3) > 10$.
- (IV) So $x > 10$ or $x + 3 > 10$.
- (V) So $x > 10$ or $x > 7$.
- (VI) Hence $x > 7$.

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Lines (I) to (III) are legal rearrangements. Line (IV) is the error: a product exceeding 10 does not mean one of its factors exceeds 10. Test $x = 3$: $3 \times 6 = 18 > 10$ so line (III) holds, but $3 > 10$ and $6 > 10$ are both false. The correct solution of $x^2 + 3x - 10 > 0$, that is $(x + 5)(x - 2) > 0$, is $x < -5$ or $x > 2$. The first invalid line is (IV): **D**.

Examiner's eye. "Product $> c$ so a factor $> c$ " is wrong for every c except in special cases. Products give an inequality only when the other side is 0: $pq > 0$ means *same sign*, not "both large".

Worked example: multiplying by an unknown sign

Question. A student solves $\frac{6}{x - 1} < 3$ for real $x \neq 1$:

- (I) Multiply both sides by $x - 1$: $6 < 3(x - 1)$.
- (II) Expand: $6 < 3x - 3$.
- (III) Rearrange: $9 < 3x$.
- (IV) Divide by 3: $x > 3$.
- (V) So the solution set is $x > 3$.
- (VI) In particular $x = 0$ is not a solution.

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Line (I) multiplies by $x - 1$, which is negative for $x < 1$; then the inequality must reverse. The line is valid only when $x > 1$. Test $x = 0$: $\frac{6}{-1} = -6 < 3$, so $x = 0$ is a solution, but $x > 3$ excludes it. The first invalid step is (I):

A. The correct solution is $x < 1$ or $x > 3$ (rearrange to $\frac{6 - 3(x - 1)}{x - 1} < 0$, that is $\frac{9 - 3x}{x - 1} < 0$, and use a sign table).

Examiner's eye. Never multiply or divide an inequality by an expression with an unknown sign. Move everything to one side and use a sign table, or split into the two cases $x > 1$ and $x < 1$.

Worked example: a correct argument that looks suspicious

Question. A student solves $|x + 1| < |x - 3|$ for real x :

- (I) Both sides are non-negative, so the inequality is equivalent to $(x + 1)^2 < (x - 3)^2$.
- (II) Expand: $x^2 + 2x + 1 < x^2 - 6x + 9$.
- (III) Subtract x^2 from both sides: $2x + 1 < -6x + 9$.
- (IV) Rearrange: $8x < 8$.
- (V) So $x < 1$.
- (VI) The solution set is $x < 1$. (For example $x = 0$: $1 < 3$ is true, and $x = 2$: $3 < 1$ is false.)

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Squaring is only dangerous when a side can be negative. Here both sides are absolute values, hence ≥ 0 , and for $A, B \geq 0$ we have $A < B \iff A^2 < B^2$. Every other line is a plain rearrangement, and the final check is correct. There is no error: **G**.

Examiner's eye. Do not hunt for an error because "there must be one". Squaring, expanding and rearranging are all legal here, so the correct answer is "no error".

Proofs: calculus, number theory, and the converse

In a proof the lines are statements about integers or functions rather than equations, so a "solution set" check is not available. Instead ask: does line $k + 1$ follow from line k alone, or does it silently need an extra fact? The usual extra facts: "this number is an integer", "this function is differentiable everywhere", "this factor is non-zero".

Worked example: an integral with a variable limit

Question. A student wishes to find $F'(1)$ for $F(x) = \int_1^{x^2} t^3 dt$:

- (I) Let $G(u) = \int_1^u t^3 dt$, so that $F(x) = G(x^2)$.
- (II) By the fundamental theorem of calculus, $G'(u) = u^3$.
- (III) So $F'(x) = G'(x^2) = (x^2)^3$.
- (IV) That is, $F'(x) = x^6$.
- (V) Hence $F'(1) = 1$.
- (VI) So the gradient of F at $x = 1$ is 1.

A line (I) B line (II) C line (III) D line (IV)
E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Lines (I) and (II) are correct. Line (III) differentiates the composite $G(x^2)$ without the chain-rule factor $2x$. The correct step is $F'(x) = G'(x^2) \cdot 2x = 2x^7$. Check by integrating first: $F(x) = \left[\frac{t^4}{4} \right]_1^{x^2} = \frac{x^8 - 1}{4}$, so $F'(x) = 2x^7$ and $F'(1) = 2$, not 1. The first error is (III): **C**.

Examiner's eye. Whenever a limit of an integral is not just x , expect a chain-rule factor: $\frac{d}{dx} \int_a^{g(x)} f(t) dt = f(g(x)) g'(x)$.

Worked example: a non sequitur about primes

Question. A student claims that $n^2 + 3n + 1$ is prime for every positive integer n :

- (I) Regard $n^2 + 3n + 1$ as a quadratic in n .
- (II) Its discriminant is $3^2 - 4 = 5$, which is not a perfect square.
- (III) So it does not factorise into brackets with integer coefficients.
- (IV) So for every positive integer n the number $n^2 + 3n + 1$ has no factors other than 1 and itself.

- (V) Hence $n^2 + 3n + 1$ is prime for every positive integer n .
 (VI) For example $n = 1, 2, 3, 4, 5$ give 5, 11, 19, 29, 41, all prime.

A line (I) B line (II) C line (III) D line (IV)
 E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Lines (I) to (III) are correct: a quadratic with integer coefficients factorises into integer brackets only if its discriminant is a perfect square. Line (IV) is the jump. “The expression cannot be factorised” is a statement about polynomials; “the number is prime” is a statement about one integer at a time. A particular value can still be composite: $n = 6$ gives $36 + 18 + 1 = 55 = 5 \times 11$. So the first error is (IV): **D**. Line (VI) is true but proves nothing.

Examiner’s eye. A good habit: whenever an argument jumps from a property of an algebraic *form* to a property of every *value*, test the first few values. The statement here fails at $n = 6$.

Worked example: proving the converse by accident

Question. A student sets out to prove: “for an integer n , n^2 is divisible by 4 if and only if n is even.”

- (I) Suppose n is even, so $n = 2k$ for an integer k .
 (II) Then $n^2 = 4k^2$, which is divisible by 4.
 (III) Conversely, suppose n^2 is divisible by 4, so $n^2 = 4m$ for an integer m .
 (IV) Then $n = \pm 2\sqrt{m}$.
 (V) So $\frac{n}{2} = \pm\sqrt{m}$.
 (VI) So $\frac{n}{2}$ is an integer, and hence n is even.

A line (I) B line (II) C line (III) D line (IV)
 E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Lines (I) and (II) prove one direction. Lines (III) to (V) are correct algebra. Line (VI) assumes what has to be shown: $\sqrt{m}$ is an integer only when m is a perfect square, and nothing so far says that. (Indeed $m = 2$ gives $\sqrt{m} = \sqrt{2}$ and $\pm 2\sqrt{2}$ is not an integer.) The argument is circular: “ $n/2$ is an integer” is the same as “ n is even”. So the first error is (VI): **F**. A valid proof of the converse uses the contrapositive: if $n = 2k + 1$ then $n^2 = 4k^2 + 4k + 1$ leaves remainder 1.

Examiner’s eye. Notice that the *statement* is true, as are all of lines (I) to (V). A faulty proof can have a true conclusion. The question is about the logic, not the truth of the final line.

Worked example: a correct proof by contradiction

Question. A student proves that $\sqrt{2} + \sqrt{3}$ is irrational:

- (I) Suppose $\sqrt{2} + \sqrt{3} = r$ where r is rational.
 (II) Then $\sqrt{3} = r - \sqrt{2}$, and squaring gives $3 = r^2 - 2r\sqrt{2} + 2$.
 (III) Rearranging, $2r\sqrt{2} = r^2 - 1$.
 (IV) Since $r > 0$ (it is a sum of positive numbers), we may divide by $2r$: $\sqrt{2} = \frac{r^2 - 1}{2r}$.
 (V) The right-hand side is rational, so $\sqrt{2}$ is rational.
 (VI) This contradicts the fact that $\sqrt{2}$ is irrational, so $\sqrt{2} + \sqrt{3}$ is irrational.

A line (I) B line (II) C line (III) D line (IV)
 E line (V) F line (VI) G the argument is correct

Options A to F say where the **first** error occurs.

Solution. Check each line. (II) is correct algebra. (III) rearranges it. In (IV) the division is the dangerous step: it is only legal if $r \neq 0$, and the student has justified this ($r > 0$). (V) and (VI) follow: a ratio of rational numbers is rational, and a contradiction means the assumption was false. There is no error: **G**.

Examiner’s eye. The only candidate for an error was the division, and the student pre-empted it by noting $r > 0$. When an argument has already addressed the danger, it is probably correct.

Habits that win marks

- Go line by line and ask: does it follow from the line above, for *every* value of the variable? Mark the first line where the answer is “no”.
- Remember that later lines can be true, and the final answer can even be right, after an earlier illegal step. The question asks for the *first error*, not the first false line.
- Put every dangerous move under suspicion: dividing, squaring, taking roots, multiplying an inequality, cancelling a bracket, differentiating an integral.
- Test a suspect step with one number: one that makes the divisor zero, one that makes an inequality sign negative, or the first few values of n .
- After a squaring step, the solution is not finished until the candidates have been checked in the original equation.
- Do not invent an error. If every move is legal (squaring two non-negative sides, dividing by a non-zero constant or a proven non-zero quantity), answer “correct”.
- For “state the error” options, make sure the sentence you choose is true about the *actual* situation (check the numbers it mentions).

Check yourself

Try these without looking back, then compare with the answers on the next page.

1. “ $\sqrt{x^2} = x$ for all real x , so $\sqrt{(-3)^2} = -3$.” Find the error.
2. “ $\frac{1}{a} > \frac{1}{b}$, so $a < b$.” Is this valid?
3. “The formula works for $n = 1, 2, 3, 4$, so it is true for all n .” What type of error?
4. “If n is prime then n is odd. 9 is odd, so 9 is prime.” What is wrong?
5. A student solves $(x - 2)(x - 3) = 6$ by writing “ $x - 2 = 6$ or $x - 3 = 6$ ”. Find the error and solve correctly.
6. A student solves $x(x - 3) = 2x$ by dividing by x to get $x - 3 = 2$, $x = 5$. What is lost?
7. “ $-3 < -1$, so $(-3)^2 < (-1)^2$.” Valid?
8. In the “proof” $a = b \Rightarrow a^2 = ab \Rightarrow a^2 - b^2 = ab - b^2 \Rightarrow (a + b)(a - b) = b(a - b) \Rightarrow a + b = b$, which step is invalid?

Answers to the self-check

1. $\sqrt{x^2} = |x|$, so $\sqrt{(-3)^2} = \sqrt{9} = 3$. The claim is only true for $x \geq 0$.
2. No. $a = 1, b = -1$: $\frac{1}{a} = 1 > -1 = \frac{1}{b}$ but $a > b$. It is valid only when a and b have the same sign.
3. Checking examples instead of proving; examples cannot establish a “for all” statement.
4. It uses the converse. “Prime $\Rightarrow$ odd” is false for $n = 2$ anyway, and “odd $\Rightarrow$ prime” is false ($9 = 3 \times 3$).
5. A product equal to 6 does not mean a factor equals 6 (that rule needs 0). Expand: $x^2 - 5x = 0$, so $x = 0$ or $x = 5$.
6. The root $x = 0$ (it satisfies $x(x - 3) = 2x$). Correct: $x^2 - 5x = 0, x(x - 5) = 0$.
7. No: $9 > 1$. Squaring preserves order only for non-negative numbers.
8. The last: cancelling $(a - b)$, which equals 0 because $a = b$.

Next step

Now do the matching topic practice set: 20 questions in TMUA style, with worked solutions. Mark it, then redo the ones you missed a week later without looking at the solutions.